Asymptotic-Preserving schemes for strongly anisotropic diffusion problems and their application to large magnetic fields in plasmas

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Context and Motivations

Large magnetic field plasma processes main caracteristics

- ☞ Particle mobilities along the magnetic fiels lines much larger than transverse ones ;
- ☞ Anisotropic medium with very different alined and transverse (with respect to the magnetic field) time scales ;
- ☞ Magnetic field magnitude non-uniform in the plasma (core/edge) with a topology altered by instabilities (ITER tokamak).

Main stream numerical methods overview

- ☞ Asymptotic models derivation : P^0 for $\varepsilon \ll 1$, ε being the dimensionless cyclotron period, to remove the most constraining scales ;
- ☞ Systematic use of coordinates adapted to the the magnetic field geometry ;
- ☞ Coupling strategy with the non reduced model P^ε (interface tracking procedure).

Path followed by the team

- ☞ Developement of asymptotic preserving methods consistant with P^ε for $\varepsilon = \mathcal{O}(1)$ and with the asimptotic model P^0 for $\varepsilon \ll 1$;
- ☞ Inconditionnal stability with respect to ε.
- ☞ Use of coordinates and meshes independant of the magnetic field geometry.

$$\begin{tikzcd}
P^{\lambda,h} \arrow[r, "h \to 0"] \arrow[d, "\lambda \to 0"] & P^{\lambda} \arrow[d, "\lambda \to 0"] \\
P^{0,h} \arrow[r, "h \to 0"] & P^{0}
\end{tikzcd}$$

FIGURE 1: AP-schemes consistancy properties.

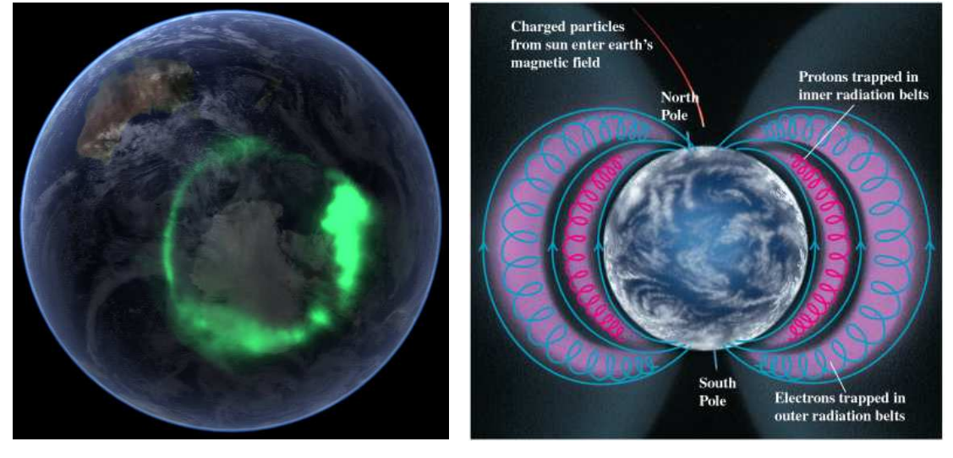


FIGURE 2: Space weather forecast : The radio waves transmission may be signigicantly altered by ionospheric plasma perturbations (aurora, particle precipitation, solar eruptions). The Earth upper atmosphere is a very anisotropic medium due to the presence of the Earth magnetic field. Left : Australis aurora (space observation). Right : Von-Allen belt (Earth magnetosphere).

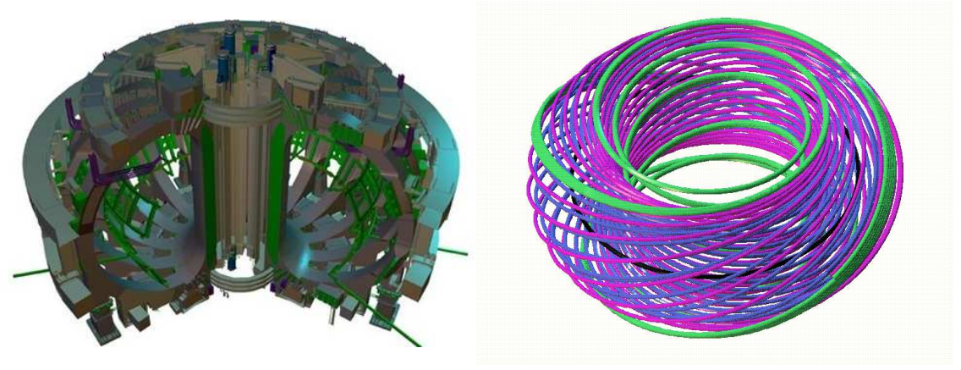


FIGURE 3: ITER : magnetically confined fusion. Left : the 48 elements of the ITER Magnet system will generate a magnetic field $2 \cdot 10^5$ times higher than that of the Earth. Right : Geometry of magnetic field lines (in the abence of instabilities).

Asymptotic preserving schemes

AP-scheme for anisotropic diffusion problems : derivation overview

☞ Definition of the model for the standard regime (P^ϵ) : the magnetic field is assumed to be aligned with the z-direction :

$$(P^\varepsilon)\quad \begin{cases} \partial^2_{xx}\phi^\varepsilon + \frac{1}{\varepsilon}\partial^2_{zz}\phi^\varepsilon = f^\varepsilon\,, & \text{in } \Omega_x \times \Omega_z\,, \\ \phi = 0\,, & \text{on } \partial\Omega_x\,, \\ \partial_z\phi = 0\,, & \text{on } \partial\Omega_z\,. \end{cases}$$

In the limit $\varepsilon \to 0$ the system P^ε degenates into

$$\begin{cases} \partial^2_{zz}\psi = 0\,, & \text{in } \Omega_z\,, \\ \partial_z\psi = 0\,, & \text{on } \partial\Omega_z\,. \end{cases}$$

This problem admits an infinte amount of solutions (all the functions of x). Consequently, standard discretizations of the P^ε problem have a conditionning number that blows up with $\varepsilon \to 0$.

☞ Integrating the anisotropic elliptic problem over Ω_z the limit of the solution $\phi^0 = \lim_{\varepsilon\to 0}\phi^\varepsilon$ verifies

$$(P^0)\quad \begin{cases} \partial^2_{xx}\phi^0 = \bar{f}\,, & \text{in } \Omega_x\,, \\ \phi^0 = 0\,, & \text{on } \partial\Omega_x\,. \end{cases}$$

where $\bar{f} = \int_{\Omega_z} f\,dz$. The problem P^0 is a well posed problem defining uniquely the solution the of P^ε in the limit $\varepsilon \to 0$.

☞ Reformulation :

AP property guarantied by the solution decomposition $\phi = \bar{\phi} + \phi'$ [DDN, SIAM10].

- The mean value $\bar{\phi}$ verifies a system similar to P^0.
- The fluctuation is provided by a system analoguous to P^ε. The property $\bar{\phi}' = 0$ ensures unicity of the solution and prevents the discrete system condition number blow up for vanishing ε (see Figure 4).

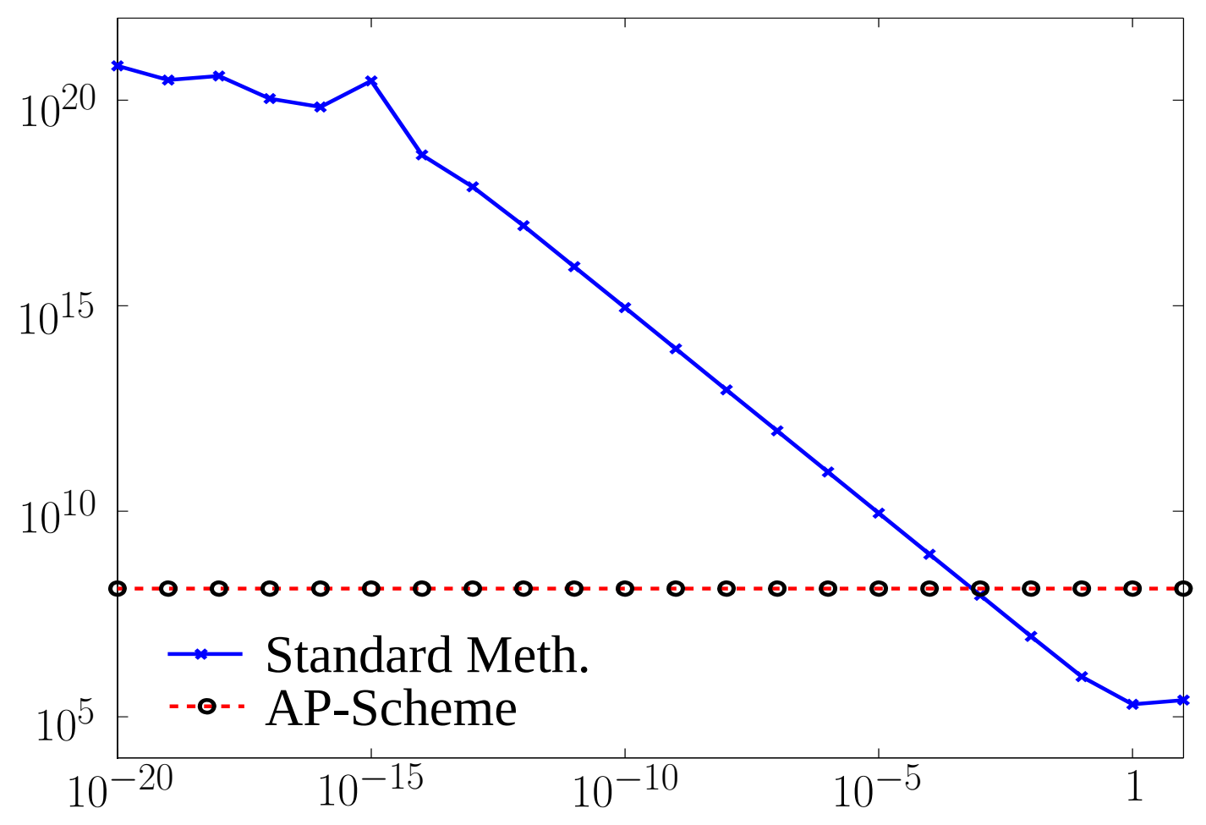


FIGURE 4: AP-scheme and P^ε standard discretization condition number as functions of ε.

AP-scheme for anisotropic diffusion problems : main achievements

☞ Extension to arbitrary anisotropy directions with Cartesian meshes and coordinates (Figure 5) :

① Duality based formulation : introduction of two Lagragian multipliers to discretize the spaces of mean and fluctuation (zero mean value) functions [DDLNN, CMS].

② Micro-macro decomposition : the number of unknowns is dramatically reduced (5 to 2) preserving the same AP-properties and accuracy [DLNN] (see Table 1).

Method	# rows	# non zero	time
Mic.-Mac.	20×10^3	623×10^3	1.156 s
Dual.Based	50×10^3	1563×10^3	7.405 s
Stand. Meth.	10×10^3	156×10^3	0.501 s

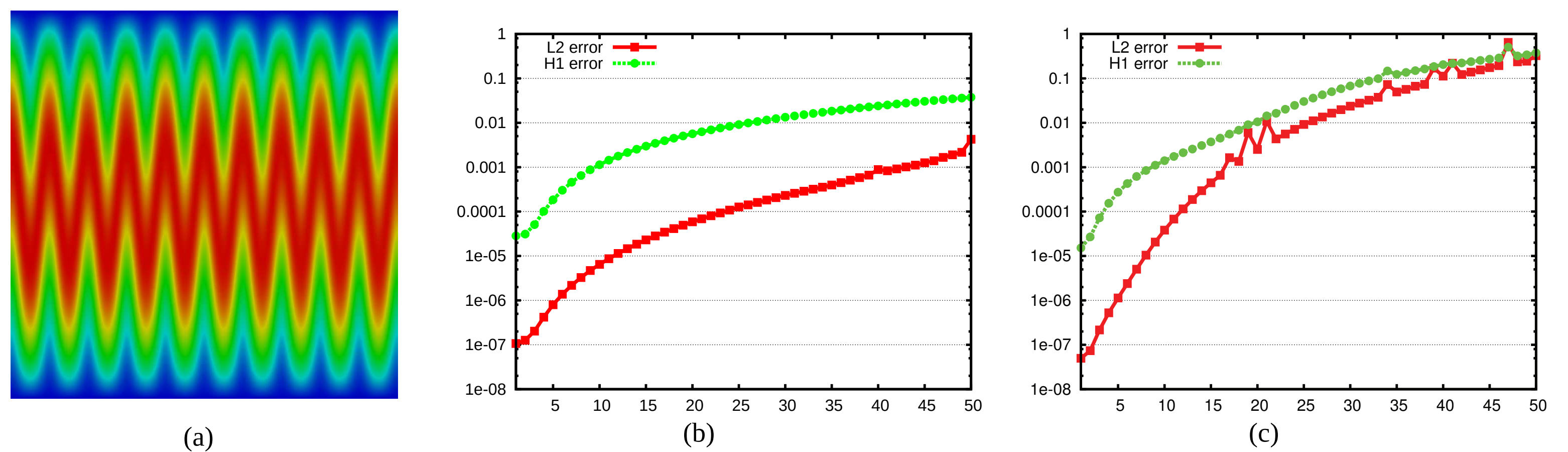
TABLE 1: Micro-Macro, Duality-Based and Standard discretizations comparison (100×100 grid).

③ Another route explored in [BDM] using a differential cracterisation of the mean and fluctuation functionnal spaces (fourth order differential problem).

☞ Application to large magnetic field plasmas : 1D Euler-Lorrentz [DDSV, JCP2009] and Valsov system [DHV] under large magnetic fields. The two dimensional Euler-Lorrentz system is investigated in [BDD, CICP]. A bifluid quasi neutral Euler-Lorentz model is considered in [BDDM, KRM] (see Figure 6).

☞ Application to non linear diffusion problems : a numerical method aimed at simulation the tokamak plasma temperature evolution is introduced in [MN] (see Figure 7). A non linear diffusion equation is investigated in [BDM] for the simulation of the full Euler-Lorentz system (with energy equation and eventual non linear internal energy laws).

Numerical simulations



(a) (b) (c)

FIGURE 5: Ellitpic anisotropic problem resolution with an heterogeneous oscillating magnetic field. (a) plot of the magnetic field as a function of the two dimensional space variable, for a frequency oscillation equal to 20. Approximation error norm for computations carried out on a uniform 400×400 Cartesian mesh with $\varepsilon = 1$ (b) and $\varepsilon = 10^{-20}$ (c) as functions of the magnetic field oscillation frequency.

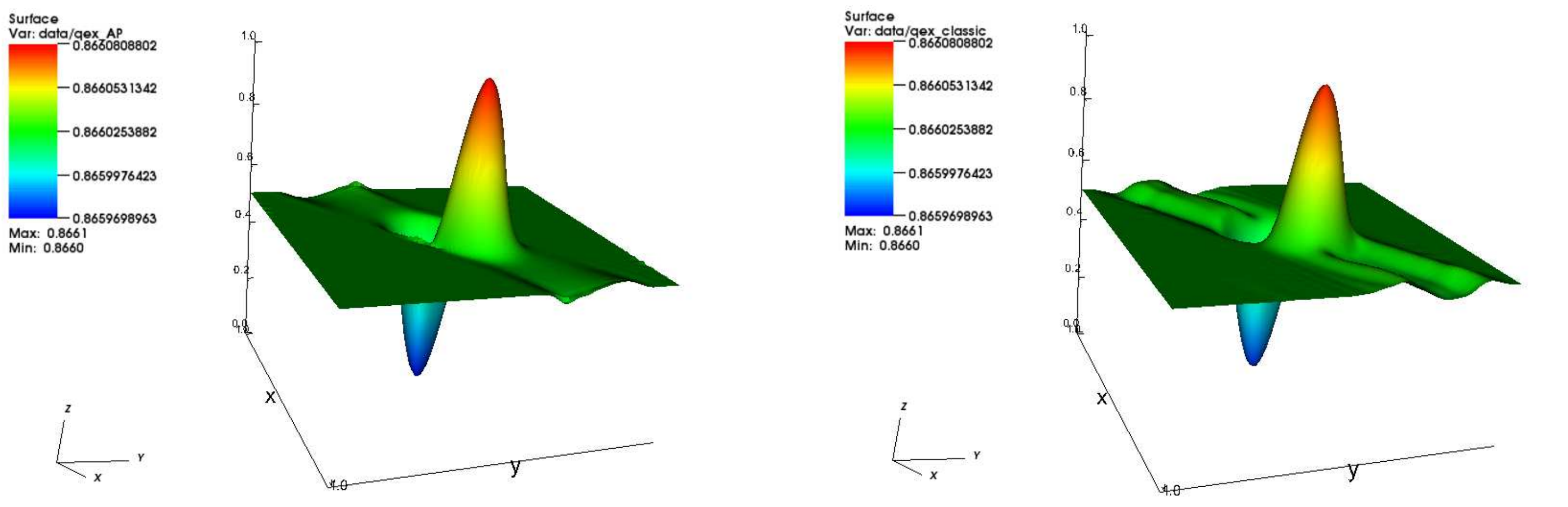


FIGURE 6: Bifluid Euler-Lorentz computations under large magnetic field and small Mach number : the dimensionless gyrperiod and the Mach number are set to 10^{-8}. Electronic momentum as a function of the two dimensional space variable. Computation carried out thanks to a classical scheme with a time step $\Delta t < 5 \cdot 10^{-9}$ (Left) and the AP-scheme with a time step $\Delta t > 10^{-6}$ (Right).

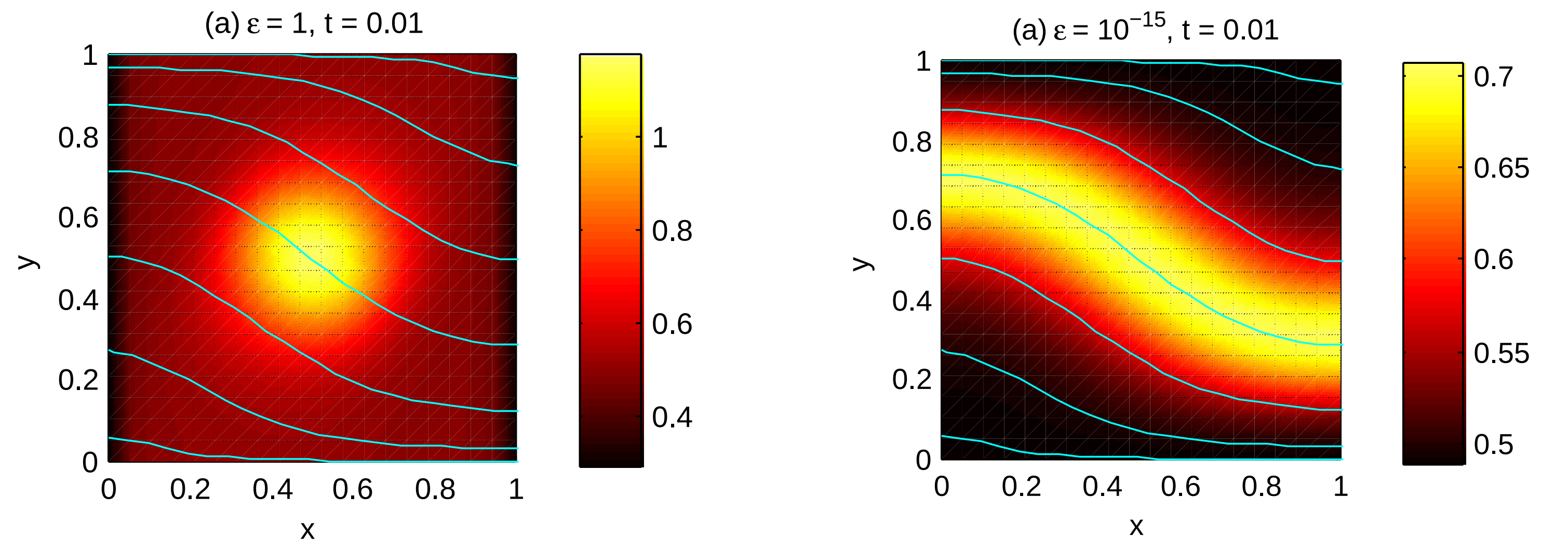


FIGURE 7: Evolution of the tokamak temperature (T) simulated by the non linear anisotropic diffusion equation : $\partial_t T - \frac{1}{\varepsilon}\nabla_{||}\cdot(K_{||}T^{5/2}\nabla_{||}T) - \nabla_\perp\cdot(K_\perp\nabla_\perp T) = 0$, with $K_{||}$, $K_\perp$ two constants and $\nabla_\perp$ the derivative along the magnetic field direction. Temperature and magnetic field lines as a function of the 2D space variable after 10^{-2} s. for $\varepsilon = 1$ (Left) and $\varepsilon = 10^{-15}$ (Right), with an isotrop gaussian as initial data.

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Acknowledgments