QPBLUR: An active-set convex QP solver

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QPBLUR: An active-set convex QP solver based on regularized KKT systems

SNOPT obtains search directions from convex QP subproblems, currently solved by SQOPT. For problems with many degrees of freedom, the nullspace active-set method of SQOPT becomes inefficient.

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QPBLUR is effectively a penalty method with bounds. QPBCL (bound-constrained Lagrangian) includes a Lagrangian term to satisfy $Ax = b$ more accurately.

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Motivation

Why another QP solver?

We would like a sparse QP solver that

- can handle a large number of free variables
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Such a solver would be useful

- inside SQP methods (like SNOPT)
- for related QPs (e.g. model predictive control)

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$$G = D(I + s_1 v_1^T)(I + s_2 v_2^T) \dots$$

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- Currently use SQOPT (active-set null-space method)

SQOPT: Large-scale convex QP

SQOPT

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Reduced-Hessian method

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Reduced-Hessian method

$$Z^T H Z \Delta x_S = -Z^T g$$

- $Z^T H Z = R^T R$ exactly
- $Z^T H Z \approx R^T R$ quasi-Newton
- CG if $dof > 2000$
- LUSOL provides reliable Basis Repair

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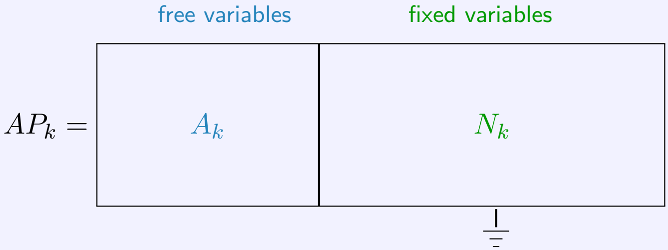

OK if A_k is nearly square:

$$10000 \times 12000 \quad \text{or} \quad 100000 \times 102000$$

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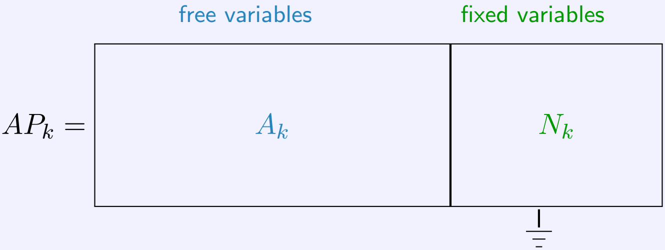

For A_k 100000×400000, we need a
QP solver based on KKT systems like QPA in GALAHAD

KKT system for current active set

$$\text{Solve } K_0 \begin{pmatrix} \Delta x_k \\ \Delta y \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

$$K_0 = \begin{bmatrix} H_k & A_k^T \\ A_k & \end{bmatrix}$$

QPBLU

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  (LUSOL, MA57, PARDISO, SuperLU, UMFPACK)
- Active-set method keeps K_0 nonsingular in theory

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work with bordered system:

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- Quasi-Newton updates to H_k handled same way
- Singular K_0 is a difficulty – need KKT repair

QPBLUR: Large-scale QP with Regularization

QPBLUR (Thesis of Chris Maes 2010)

$$\min_{x,y} \quad c^T x + \tfrac{1}{2}x^T H x + \tfrac{1}{2}\delta\|x\|_2^2 + \tfrac{1}{2}\mu\|y\|_2^2$$

$$Ax + \mu y = b, \qquad l \le x \le u$$

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- Can use Hanh's block-LU updates without change

QPBLUR strategy

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MATLAB implementation

- Scale problem

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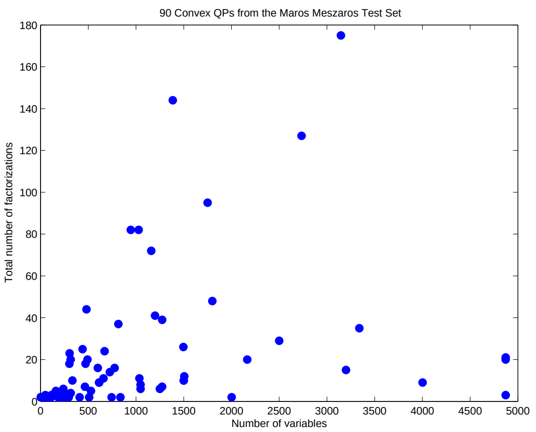
- Exit if small relative change in obj

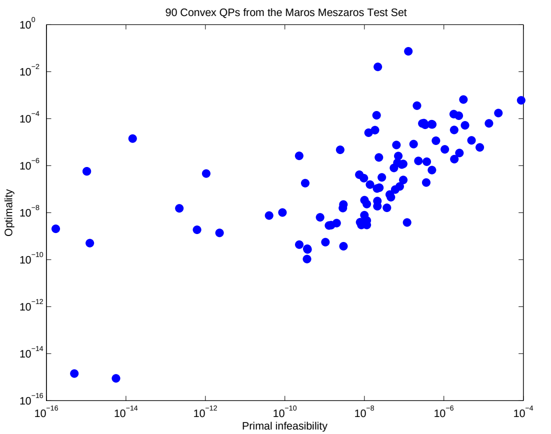
Numerical Results on 90 QP problems (Meszaros set)

KKT factorizations



No. of factorizations on 90 Meszaros QP test problems

Accuracy of solutions



Residuals for 90 Meszaros QP test problems

QPBLUR pros and cons

$$\min_{x,y}\ c^Tx + \tfrac{1}{2}x^THx + \tfrac{1}{2}\delta x^Tx + \tfrac{1}{2}\mu y^Ty \qquad Ax + \mu y = b,\ l \le x \le u$$

Advantages

- Warm starts (any x_0, any working set)

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- Tiny regularization: risks ill-conditioned KKT (but so far so good)

Penalty vs Augmented Lagrangian

Rethink QPBLUR

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$\equiv$ quadratic penalty function:

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Why not Augmented Lagrangian?:

$$\min_{\ell \le x \le u} \ \phi(x) - \hat{y}^T(b - Ax) + \tfrac{1}{2}\rho \|b - Ax\|_2^2$$

BCL method as in LANCELOT

BCL Method (LANCELOT)

$$\min\ \phi(x)\ \ \text{st}\ \ c(x) = 0,\quad \ell \le x \le u$$

Bound-constrained augmented Lagrangian method

$$L(x, \hat{y}, \rho) = \phi(x) - \hat{y}^T c(x) + \tfrac{1}{2}\rho \|c(x)\|^2$$

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- Otherwise, increase ρ
- Repeat

BCL Method for QP

$$\min_{x} \ \phi(x) \ \ \text{st} \ \ Ax = b, \ \ \ell \le x \le u$$

BCL subproblem

$$\begin{aligned} \min_{x,\,r} \ L(x, \hat{y}, \rho) \quad &= \quad \phi(x) + \hat{y}^T r + \tfrac{1}{2}\rho\|r\|^2 \\ & \quad\ \ Ax + r = b, \qquad \ell \le x \le u \end{aligned}$$

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- Solve subproblem to get $\hat{x}$, $\hat{r}$ (optimality tol $\omega \to 0$)
- If $\|\hat{r}\| < \eta$, update $\hat{y}$ (feasibility tol $\eta \to 0$)
- Otherwise, increase ρ

BCL Method for QP

$$\min_x \ \phi(x) \ \text{ st } \ Ax = b, \ \ \ell \le x \le u$$

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- Repeat

QPBCL

QPBCL: Active-set method for convex QP

$$AP = \begin{pmatrix} A_k & N_k \end{pmatrix}$$

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- Same system as QPBLUR – just different rhs
- Still single phase
- Still use LUSOL, MA57, UMFPACK, ...
- Still use Hanh's block-LU updates

Numerical Results

90 Meszaros QP problems

QPBCL solves with smaller penalty ρ (hence larger $\mu \equiv 1/\rho$) compared to QPBLUR

COPS 3.0 problems with many degrees of freedom

	m	n	*dof*
dirichlet	42	8981	5355
henon	82	10801	9410
lane_emden	82	19240	5414
minsurf	0	5000	4782

- SNOPT/AMPL
- Major iterations generate QPs, solved by SQOPT
- QP data loaded into MATLAB, solved by QPBCL

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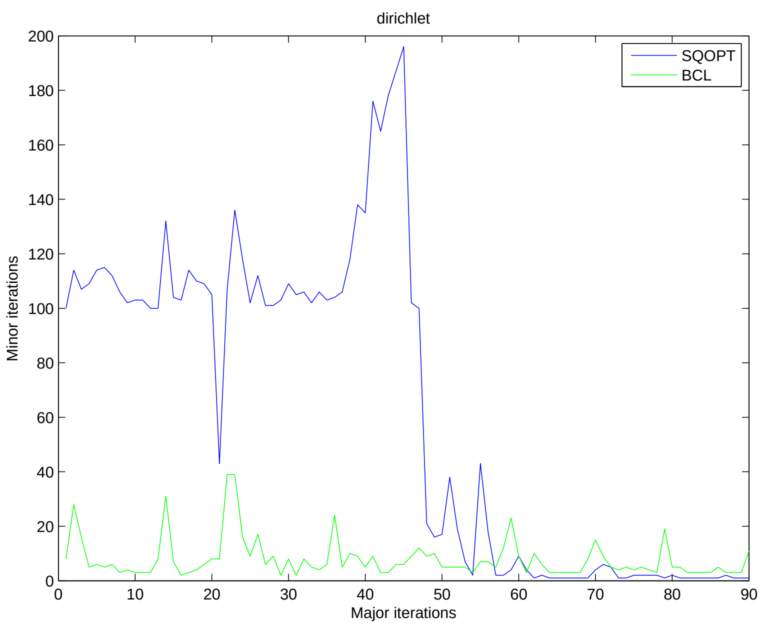
Compare SQOPT iterations and QPBCL iterations

There are 3 kinds of people: Those who can count, and those who cannot.

— George Carlin


dirichlet
Minor iterations
Major iterations
SQOPT
BCL
0
20
40
60
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100
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160
180
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0
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70
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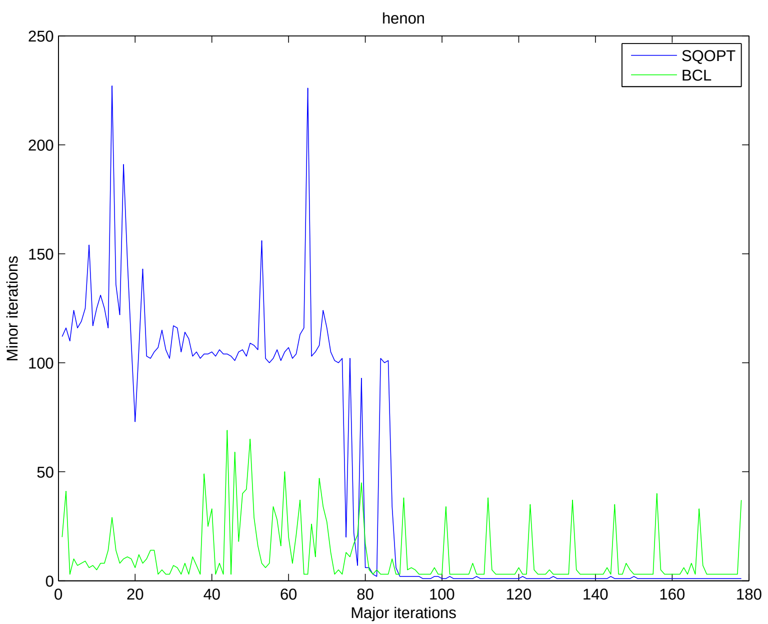
90


henon
SQOPT
BCL
Minor iterations
Major iterations
0
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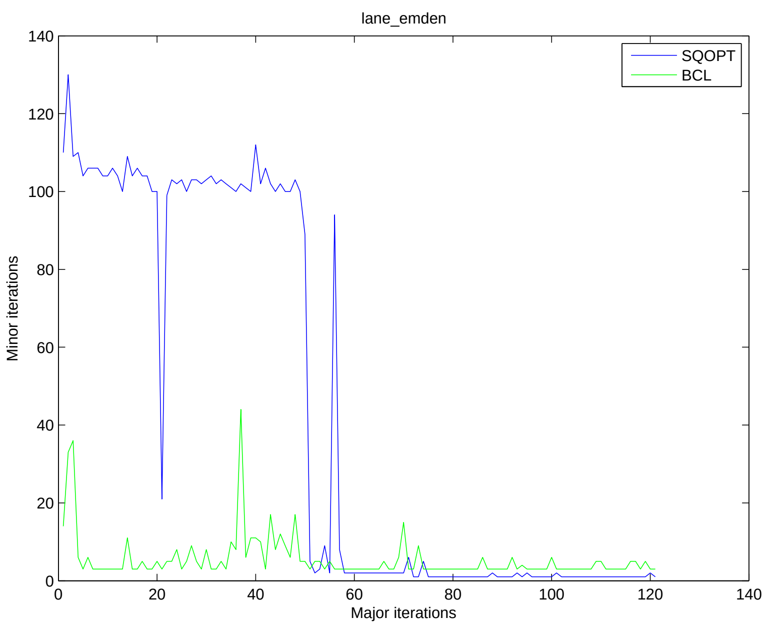
180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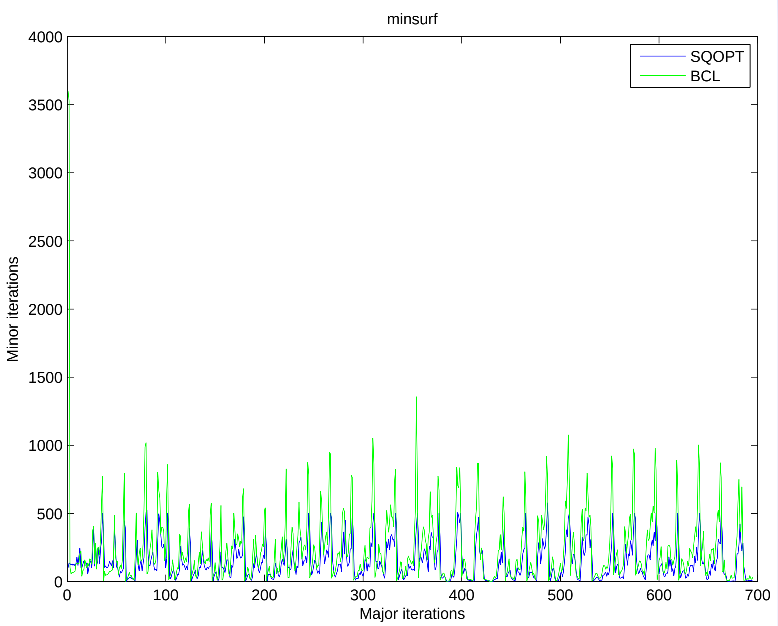
minsurf
4000
3500
3000
2500
2000
1500
1000
500
0
Minor iterations
SQOPT
BCL
0
100
200
300
400
500
600
700
Major iterations

Augmented Lagrangian methods for QP

- Conn, Gould, and Toint (1992)
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- C. Kirches, H. G. Bock, J. P. Schlöder and S. Sager (2009)
  A factorization with update procedures for a KKT matrix arising in direct optimal control
  Active-set method with block-structured KKT factorization

Even convex QP isn't easy

but conferences help us make progress

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Brigitte Yzel

STAE
RTRA
CERFACS