Optimal control of descriptor systems

Volker Mehrmann
Institut für Mathematik
Technische Universität Berlin

With J. Heiland, P. Kunkel, M. Schmidt

DFG Research Center MATHEON
Mathematics for key technologies

Outline

Consider descriptor systems (differential-algebraic equations, DAEs)

$$F(t, x, u, \dot{x}, p, \omega) = 0, \;\; x(\underline{t}) = \underline{x}, t \in [\underline{t}, \overline{t}]$$

- x–state, u–input, p–parameters, ω– uncertainties.
- Often one also considers outputs $y = g(t, x, u)$.
- One is typically interested in feedback controllers (closed loop) $u(t) = k(x(t))$, or better $u(t) = k(y(t))$.

Closed loop vs. open loop

- In open loop control, we determine the control before and then apply it dynamically.
- In closed loop control one reacts on the current state $u(t) = k(x(t))$ or one partially observes the system by an output y and react on this $u(t) = k(y(t))$.
- Closed loop control allows to react on disturbances, model or computational errors.
- For fast systems or systems with uncertainty, closed loop is essential.
- Typical tasks in closed loop control: Stabilization of system that has become unstable or following a reference trajectory.
- Usually there is still freedom in the choice of the control u which is typically fixed by minimizing a cost or energy functional: optimal control.

Optimal control problem:

$$\mathcal{J}(x,u) = \mathcal{M}(x(\bar{t})) + \int_{\underline{t}}^{\bar{t}} \mathcal{K}(t,x,u)\,dt = \min!$$

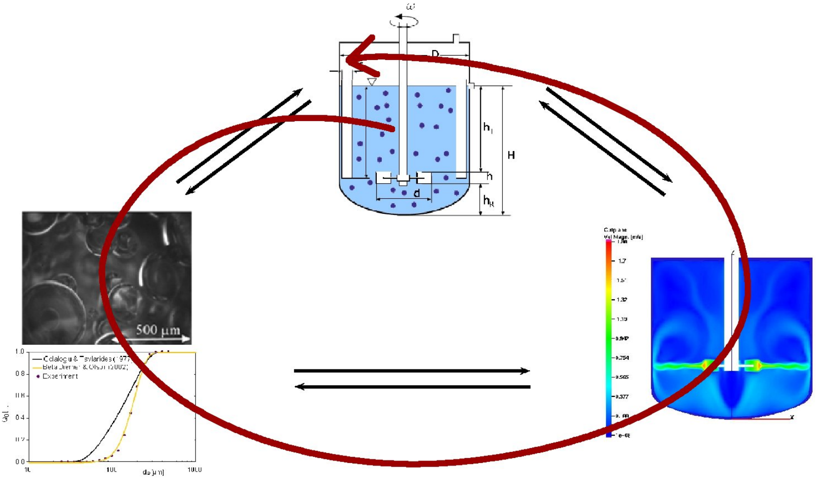
subject to the DAE constraint

$$F(t,x,u,\dot{x}) = 0,\;\; x(\underline{t}) = \underline{x}.$$

Outline

Drop size distributions

with S. Schmelter and M. Kraume (Chemical Eng., TU Berlin)



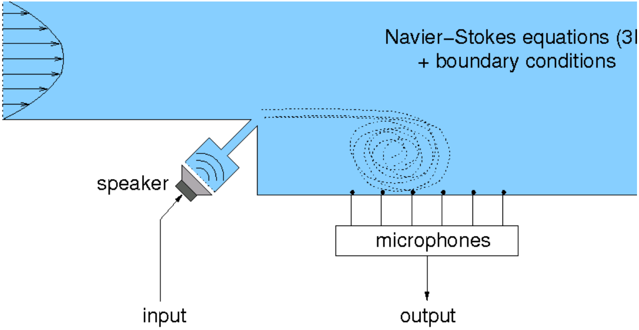
Technological Application, Tasks

Chemical industry: pearl polymerization and extraction processes

- Modelling of coalescence and breakage in turbulent flow.
- Numerical methods for coupled system of population balance equations/fluid flow equations.
- Development of closed loop control methods.
- Model reduction and observer design.
- Control of real configurations via stirrer speed.

Goal: Achieve specified average drop diameter and small standard deviation by real time-control of stirrer-speed.

Project in collaborative research centre 557 at TU Berlin *Control of complex shear flows*, (with F. Tröltzsch)



Technological Application, Tasks

Control of detached turbulent flow on airline wing

- Test case: move recirculation bubble in backward step.
- Modelling of turbulent flow.
- Development of control methods for large scale coupled systems.
- Model reduction and observer design.
- Optimal control of real configurations via blowing and sucking of air in wing.

Ultimate goals: Force detached flow back to wing, control the flow field behind the airplane.

Other applications

- Control of automatic gearboxes (Project with Daimler AG)
- Electrical circuit simulation and control (project with NEC Europe)
- Model reduction for electrical and acoustic field computations (projects with CST GmbH and SFE GmbH)
- Control of vibrating structures.
- . . ..

Outline

After space discretization all these applications lead to DAE control systems

$$\mathcal{F}(t, x, \dot{x}, u) = 0,$$

or in the linear case (linearization along solutions)

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t).$$

For the mathematical analysis we can use a behavior approach, i.e., forming $z = (x, u)$ ($z = (x, u, y)$) we obtain general non-square DAEs

$$F(t, z, \dot{z}) = 0, \quad \mathcal{E}(t)\dot{z} = \mathcal{A}(t)z.$$

Why DAEs and not ODEs?

DAEs provide a unified framework for the analysis, simulation and control of coupled dynamical systems (continuous and discrete time).

- Automatic modelling leads to DAEs (Constraints at interfaces). This is standard in electrical, mechanical and chemical engineering. SIMULINK, SPICE, DYMOLA, ...
- Conservation laws lead to DAEs after space discretization. (Conservation of mass, energy, volume, momentum).
- Coupling of multiphysical systems leads to DAEs.
- Coupling of solvers leads to DAEs (discrete time).
- Control problems are DAEs (behavior).
- DAEs allow to incorporate state constraints.

Black-box modelling with DAEs

Modelling becomes extremely easy with DAEs, but:

- ▷ Numerical simulation does not always work, instabilities and convergence problems occur (e.g. SIMULINK) !
- ▷ Consistent initialization is difficult.
- ▷ More regularity (smoothness) necessary than it seems. Solution depends on derivatives of the state equations.
- ▷ Numerical drift-off phenomenon due to hidden constraints.
- ▷ Model reduction is difficult.
- ▷ Classical control approaches are difficult (non-proper transfer functions).

Black-box DAE modelling pushes all difficulties into the mathematical methods. In general the analytic and numerical methods cannot handle this!

How does one solve this today in practice?

- Simplified models and remodelling.
- Space discretization with coarse meshes.
- Identification and realization of black box models.
- Space discretization of fine model followed by model reduction (mostly based on heuristic methods).
- Use of standard optimal control techniques for simplified mathematical model.
- But do they work for these models?

Is there anything to do ?

Why not just apply the classical Pontryagin maximum principle?

- ▷ The solution of DAE problems may depend on derivatives of the inputs.
- ▷ It is very difficult to find the right adjoint equations.
- ▷ DAEs may not be (uniquely) solvable for all $u \in \mathbb{U}$ and all f and the initial conditions are restricted.
- ▷ For many years there was no maximum principle available.
- ▷ DAEs model state constraints and thus all the difficulties with state constraints occur.

- Linear constant coefficient d-index 1 case, Bender/Laub 87, Campbell 87, M. 91, Geerts 93.
- Regularization to d-index 1, Bunse-Gerstner/M./Nichols 94, Byers/Geerts/M. 97, Byers/Kunkel/M. 97.
- Linear variable coefficients d-index 1 case, Kunkel./M. 97.
- Semi-explicit nonlinear d-index 1 case, maximum principle, De Pinho/Vinter 97, Devdariani/Ledyaev 99.
- Semi-explicit d-index $2, 3$ case Roubicek/Valasek 02.
- Linear d-index $1, 2$ case with properly stated leading term, Balla/März, 02,04, Balla/Linh 05, Kurina/März 04, Backes 06.
- Multibody systems (structured and of d-index 3), Büskens/Gerdts 00, Gerdts 03,06.
- General linear and nonlinear case Kunkel/M. 08.

For an appropriate remodelling, for the numerical solution of general DAEs and for the design of controllers, we need derivatives. Derivative arrays (Campbell 89).

We assume that derivatives of original functions can be obtained via computer algebra or automatic differentiation.

Linear case: We put $E(t)\dot{x} = A(t)x + f(t)$ and its derivatives up to order μ into a large DAE

$$M_k(t)\dot{z}_k = N_k(t)z_k + g_k(t), \quad k \in \mathbb{N}_0$$

for $z_k = (x, \dot{x}, \ldots, x^{(k)})$.

$$M_2 = \begin{bmatrix} E & 0 & 0 \\ A - \dot{E} & E & 0 \\ \dot{A} - 2\ddot{E} & A - \dot{E} & E \end{bmatrix}, \; N_2 = \begin{bmatrix} A & 0 & 0 \\ \dot{A} & 0 & 0 \\ \ddot{A} & 0 & 0 \end{bmatrix}, \; z_2 = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}.$$

Theorem (Kunkel/M. 96)

Under some constant rank assumptions, for a linear DAE there exist integers μ, a, d and v such that:

1. $\operatorname{corank} M_{\mu+1}(t) - \operatorname{corank} M_{\mu}(t) = v$.
2. $\operatorname{rank} M_{\mu}(t) = (\mu+1)m - a - v$ *on* $\mathbb{I}$, *and there exists a smooth matrix function* $Z_{2,3}$ *(left nullspace of* M_{μ}*) with* $Z_{2,3}^T M_{\mu}(t) = 0$.
3. *The projection* $Z_{2,3}$ *can be partitioned into two parts:* Z_2 *(left nullspace of* $[M_{\mu}, N_{\mu}]$*) so that the first block column* $\hat{A}_2$ *of* $Z_2^* N_{\mu}(t)$ *has full rank* a *and* $Z_3^* N_{\mu}(t) = 0$. *Let* T_2 *be a smooth matrix function such that* $\hat{A}_2 T_2 = 0$, *(right nullspace of* $\hat{A}_2$*).*
4. $\operatorname{rank} E(t) T_2 = d = l - a - v$ *and there exists a smooth matrix function* Z_1 *of size* (n, d) *with* $\operatorname{rank} \hat{E}_1 = d$, *where* $\hat{E}_1 = Z_1^T E$.

Reduced problem

- The quantity μ is called the strangeness-index. It describes the smoothness requirements for forcing or input functions.
- It generalizes the differentiation (d-)index to over- and underdetermined DAEs (and counts differently).
- We obtain a **numerically computable** reduced system:

$$
\begin{array}{rclr}
\hat{E}_1(t)\dot{x} & = & \hat{A}_1(t)x + \hat{f}_1(t), & d \text{ differential equations} \\
0 & = & \hat{A}_2(t)x + \hat{f}_2(t), & a \text{ algebraic equations} \\
0 & = & \hat{f}_3(t), & v \text{ consistency equations}
\end{array}
$$

where $\hat{A}_1 = Z_1^T A$, $\hat{f}_1 = Z_1^T f$, and $\hat{f}_2 = Z_2^T g_\mu$, $\hat{f}_3 = Z_3^T g_\mu$.

- The reduced system has the same solution set as the original problem but now it has strangeness-index 0. Remodelling!
- **We assume from now on this reduced system.**

Hypothesis: There exist integers μ, r, a, d, and v such that $\mathbf{L} = F_\mu^{-1}(\{0\}) \neq \emptyset$.
We have $\operatorname{rank} F_{\mu;t,x,\dot{x},\ldots,x^{(\mu+1)}} = \operatorname{rank} F_{\mu;x,\dot{x},\ldots,x^{(\mu+1)}} = r$, in a neighborhood of $\mathbf{L}$ such that there exists an equivalent system $\tilde{F}(z_\mu) = 0$ with a Jacobian of full row rank r. On $\mathbf{L}$ we have

1. $\operatorname{corank} F_{\mu;x,\dot{x},\ldots,x^{(\mu+1)}} - \operatorname{corank} F_{\mu-1;x,\dot{x},\ldots,x^{(\mu+1)}} = v$.
2. $\operatorname{corank} \tilde{F}_{x,\dot{x},\ldots,x^{(\mu+1)}} = a$ and there exist smooth matrix functions Z_2 (left nullspace of M_μ) and T_2 (right nullspace of $\hat{A}_2 = \tilde{F}_x$) with $Z_2^T \tilde{F}_{x,\dot{x},\ldots,x^{(\mu+1)}} = 0$ and $Z_2^T \hat{A}_2 T_2 = 0$.
3. $\operatorname{rank} F_{\dot{x}} T_2 = d, \quad d = \ell - a - v$, and there exists a smooth matrix function Z_1 with $\operatorname{rank} Z_1^T F_{\dot{x}} = d$.

Theorem (Kunkel/M. 02)

The solution set $\mathbf{L}$ *forms a (smooth) manifold of dimension* $(\mu+2)n+1-r$.
The DAE can locally be transformed (by application of the implicit function theorem) to a reduced DAE of the form

$$\begin{aligned} \dot{x}_1 &= \mathcal{L}(t, x_1, x_3), && \textit{(d differential equations)}, \\ x_2 &= \mathcal{K}(t, x_1, x_3), && \textit{(a algebraic equations)}, \\ 0 &= 0 && \textit{(v redundant equations)}. \end{aligned}$$

The variables x_3 *represent undetermined components (controls).*

Ideally we would like to get the model in this form directly, otherwise this has to be computed at each time point.
We typically use the structure to get this cheaply.

We get consistent initial values by solving $F_\mu(t_0, x, \dot{x}, \ldots, x^{(\mu+1)}) = 0$ at t_0 for the algebraic variable $(x, \dot{x}, \ldots, x^{(\mu+1)})$.
For the numerical integration of the DAE, e.g. with BDF, the system

$$\begin{aligned} F_\mu(t_i + h, x, \dot{x}, \ldots, x^{(\mu+1)}) &= 0, \\ \tilde{Z}_1^T F(t_i + h, x, D_h x) &= 0 \end{aligned}$$

is solved for the algebraic variable $(x_i, \dot{x}_i, \ldots, x^{(\mu+1)_i})$ at $t_i + h$.
Here, $\tilde{Z}_1$ is an approximation of Z_1 at the desired solution, and

$$D_h x_i = \frac{1}{h} \sum_{l=0}^{k} \alpha_l x_{i-l},$$

is the discretization by BDF or other finite difference operators.

Theorem (Kunkel/M. 2002)

Let F satisfy the nonlinear Hypothesis.
Then, the occurring Jacobians of the system have full row rank at the solution provided the step-size h is sufficiently small and the approximation $\tilde{Z}_1$ is sufficiently good.

- *Simplified Gauss-Newton method can be used to solve the nonlinear systems at every integration step.*
- *The order and convergence properties are as for ODEs.*
- *Method can be implemented by using local computations only.*

Outline

Cost functional:

$$\mathcal{J}(x,u) = \frac{1}{2}x(\bar{t})^T M x(\bar{t}) + \frac{1}{2}\int_{\underline{t}}^{\bar{t}} (x^T W x + 2x^T S u + u^T R u)\, dt,$$

$W = W^T \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $S \in C^0(\mathbb{I}, \mathbb{R}^{n,l})$, $R = R^T \in C^0(\mathbb{I}, \mathbb{R}^{l,l})$, $M = M^T \in \mathbb{R}^{n,n}$.

Constraint:

$$E(t)\dot{x} = A(t)x + B(t)u + f(t), \quad x(\underline{t}) = \underline{x},$$

$E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $A \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B \in C^0(\mathbb{I}, \mathbb{R}^{n,l})$, $f \in C^0(\mathbb{I}, \mathbb{R}^{n})$, $\underline{x} \in \mathbb{R}^n$.

Here: Determine optimal controls $u \in \mathbb{U} = C^0(\mathbb{I}, \mathbb{R}^l)$.

More general spaces, output controls, and also nonsquare E, A are possible.

Calculus of variations, linear ODEs (E=I)

Introduce Lagrange multiplier function $\lambda(t)$ and couple constraint into cost function, i.e. minimize

$$\begin{aligned}\tilde{\mathcal{J}}(x,u,\lambda) &= \frac{1}{2}x(\bar{t})^T M x(\bar{t}) + \frac{1}{2}\int_{\underline{t}}^{\bar{t}} (x^T W x + 2x^T S u + u^T R u \\ &+ \lambda^T(\dot{x} - Ax + Bu + f)\, dt.\end{aligned}$$

Consider variations $x + \delta x$, $u + \delta u$ and $\lambda + \delta\lambda$.
For a minimum the cost function has to go up in the neighborhood, so we get optimality conditions (Euler-Lagrange equations):

Theorem

If (x, u) is a solution to the optimal control problem, then there exists a Lagrange multiplier function $\lambda \in C^1(\mathbb{I}, \mathbb{R}^n)$, such that (x, λ, u) satisfy the optimality boundary value problem

$$
\begin{array}{ll}
\text{(a)} & \dot{x} = Ax + Bu + f, \;\; x(\underline{t}) = \underline{x}, \\
\text{(b)} & \dot{\lambda} = Wx + Su - A^T\lambda, \;\; \lambda(\bar{t}) = -Mx(\bar{t}), \\
\text{(c)} & 0 = S^T x + Ru - B^T\lambda.
\end{array}
$$

Naive Idea for DAEs

Replace the identity in front of x by $E(t)$ and then do the analysis in the same way.
For DAEs the formal optimality system could be

$$
\begin{array}{llcl}
\text{(a)} & E\dot{x} & = & Ax + Bu + f, \; x(\underline{t}) = \underline{x} \\
\text{(b)} & \frac{d}{dt}(E^T\lambda) & = & Wx + Su - A^T\lambda, \; (E^T\lambda)(\bar{t}) = -Mx(\bar{t}), \\
\text{(b)} & 0 & = & S^Tx + Ru - B^T\lambda.
\end{array}
$$

- In general not true. Counterexamples: Backes 2006, Kunkel/M. 2008
- One has to guarantee that the resulting adjoint equation for λ has a unique solution, but it may not.
- The formal boundary conditions may not be consistent.

To derive optimality conditions for DAEs, we need the right solution space for x. (Recall that we are in the reduced case.)

$$\mathbb{X} = C^1_{E^+E}(\mathbb{I}, \mathbb{R}^n) = \left\{x \in C^0(\mathbb{I}, \mathbb{R}^n) \mid E^+Ex \in C^1(\mathbb{I}, \mathbb{R}^n)\right\},$$

where E^+ denotes the Moore-Penrose inverse of the matrix valued function $E(t)$, i.e. the unique matrix function that satisfies the Penrose axioms.

$$EE^+E = E, E^+EE^+ = E^+, (EE^+)^T = EE^+, (E^+E)^T = E^+E$$

The input space $\mathbb{U}$ is usually a set of piecewise continuous functions (or a space of distributions.)

Theorem (Kunkel/M. 08)

Consider the linear quadratic DAE optimal control problem with a consistent initial condition. Suppose that the system has $\mu = 0$ *as a behavior system and that* $Mx(\bar{t}) \in \operatorname{cokernel} E(\bar{t})$.
If $(x, u) \in \mathbb{X} \times \mathbb{U}$ *is a solution to this optimal control problem, then there exists a Lagrange multiplier function* $\lambda \in C^1_{E^+E}(\mathbb{I}, \mathbb{R}^n)$, *such that* (x, λ, u) *satisfy the optimality boundary value problem*

$$
\begin{aligned}
&E\tfrac{d}{dt}(E^+Ex) = (A + E\tfrac{d}{dt}(E^+E))x + Bu + f, \;\; (E^+Ex)(\underline{t}) = \underline{x},\\
&E^T\tfrac{d}{dt}(EE^+\lambda) = Wx + Su - (A + EE^+\dot{E})^T\lambda,\\
&(EE^+\lambda)(\bar{t}) = -E^+(\bar{t})^T Mx(\bar{t}),\\
&0 = S^Tx + Ru - B^T\lambda.
\end{aligned}
$$

Remarks

- If a minimum exists, then it satisfies the optimality system.
- If a unique solution to the formal optimality system exists, then x, u are the same, λ may be different.
- The optimality DAE may have $\mu > 0$. Then further consistency conditions or smoothness requirements arise.
- The condition that the original system has $\mu = 0$ as a behavior system is not necessary if the cost function is chosen appropriately.
- Under some extra conditions (invertibility of the weight matrix R, etc) the solution is a feedback control (Riccati).
- In general one has to solve the optimality boundary value problem.

Reduced strangeness-free system

Recall that the DAE can locally be transformed to a reduced DAE of the form

$$\begin{aligned} \dot{x}_1 &= \mathcal{L}(t, x_1, u), && (d \text{ differential equations}), \\ x_2 &= \mathcal{K}(t, x_1, u), && (a \text{ algebraic equations}), \\ 0 &= 0 && (v \text{ redundant equations}). \end{aligned}$$

Theorem (Kunkel/M. 2008)

Consider the nonlinear optimal control problem and assume that $\mu = 0$ for the system in behavior form, then in terms of the reduced DAE, the local optimality system is

$$
\begin{aligned}
&\text{(a)} \quad \dot{x}_1 = \mathcal{L}(t, x_1, u), \ x_1(\underline{t}) = \underline{x}_1, \\
&\text{(b)} \quad x_2 = \mathcal{R}(t, x_1, u), \\
&\text{(c)} \quad \dot{\lambda}_1 = \mathcal{K}_{x_1}(t, x_1, x_2, u)^T - \mathcal{L}_{x_1}(t, x_1, x_2, u)^T \lambda_1 - \mathcal{R}_{x_1}(t, x_1, u)^T \lambda_1 \\
&\qquad\qquad \lambda_1(\bar{t}) = -\mathcal{M}_{x_1}(x_1(\bar{t}), x_2(\bar{t}))^T \\
&\text{(d)} \quad 0 = \mathcal{K}_{x_2}(t, x_1, x_2, u)^T + \lambda_2, \\
&\text{(e)} \quad 0 = \mathcal{K}_u(t, x_1, x_2, u)^T - \mathcal{L}_u(t, x_1, u)^T \lambda_1 - \mathcal{R}_u(t, x_1, u)^T \lambda_2, \\
&\text{(f)} \quad \gamma = \lambda_1(\underline{t})
\end{aligned}
$$

Here λ_1, λ_2 are Lagrange multipliers associated with x_1, x_2 and γ is associated with the initial value constraint.

- These are local results.
- All the results can be generalized to general nonsquare nonlinear systems.
- End point conditions for x can be included.
- Input and state inequality constraints can be included to give a maximum principle.

Linear case: Given $E(t), A(t), B(t), f(t)$ in the DAE and $S(t), R(t), W(t), M$ from the cost functional.
The resulting linear optimality system has the form

$$
\begin{array}{ll}
\text{(a)} & \hat{E}_1 \dot{x} = \hat{A}_1 x + \hat{B}_1 u + \hat{f}_1, \ (\hat{E}_1^+ \hat{E}_1 x)(\underline{t}) = \underline{x} \\
\text{(b)} & 0 = \hat{A}_2 x + \hat{B}_2 u + \hat{f}_2, \\
\text{(c)} & \frac{d}{dt}(\hat{E}_1^T \lambda_1) = Wx + Su - \hat{A}_1^T \lambda_1 - \hat{A}_2^T \lambda_2, \\
 & \qquad \lambda_1(\bar{t}) = -[\, \hat{E}_1^+(\bar{t})^T \ \ 0 \,] M x(\bar{t}), \\
\text{(d)} & 0 = S^T x + Ru - \hat{B}_1^T \lambda_1 - \hat{B}_2^T \lambda_2.
\end{array}
$$

where $\hat{E}_i, \hat{A}_i, \hat{B}_i, \hat{f}_i$ are obtained by projection with smooth orthogonal projections Z_i from the derivative array.
An analogous structure arises locally in the nonlinear case.

Numerical Problems

- In the implementation of our numerical integration codes we may use nonsmooth projectors Z_1^T, Z_2^T, since it would be too expensive to carry smooth projectors along.
- For numerical forward (in time) simulation, it is enough that we know the existence of smooth projectors.
- Integration methods like Runge-Kutta or BDF do not see the nonsmooth behavior.
- But the adjoint variables (Lagrange multipliers) depend on these projections and their derivatives.

However, even if Z_1^T, Z_2^T are nonsmooth, $Z_1 Z_1^T$ and $Z_2 Z_2^T$ are smooth.

- Choose

$$\hat{E}_1^T \lambda_1 = E^T Z_1 \lambda_1 = E^T Z_1 Z_1^T Z_1 \lambda_1 = E^T Z_1 Z_1^T \hat{\lambda}_1.$$

- With $\hat{\lambda}_1 = Z_1 \lambda_1$ we obtain smooth coefficients for $\hat{\lambda}_1$.
- However, we have to add the condition that $\hat{\lambda}_1 \in \mathrm{range}\, Z_1$ to the system.
- If Z_i' completes Z_i to a full orthogonal matrix (we compute these anyway when doing a QR or SVD computation) then these conditions can be expressed as

$$Z_i'^T \hat{\lambda}_i = 0, \ i = 1, 2$$

New linear optimality system

For the numerical solution we use the optimality system.

$$
\begin{array}{ll}
\text{(a)} & \hat{E}_1 \dot{x} = \hat{A}_1 x + \hat{B}_1 u + \hat{f}_1, \ (\hat{E}_1^+ \hat{E}_1 x)(\underline{t}) = \underline{x}, \\
\text{(b)} & 0 = \hat{A}_2 x + \hat{B}_2 u + \hat{f}_2, \\
\text{(c)} & \frac{d}{dt}(E^T Z_1 Z_1^T \hat{\lambda}_1) = Wx + Su - A^T \hat{\lambda}_1 - [[0 I | 00 | \cdots | 00] N_\mu^T \hat{\lambda}_2, \\
 & \qquad (Z_1^T \hat{\lambda}_1)(\bar{t}) = -[\ \hat{E}_1^+(\bar{t})^T \ \ 0\ ] M x(\bar{t}), \\
\text{(d)} & 0 = S^T x + Ru - B^T \hat{\lambda}_1 - [0 I | 00 | \cdots | 00] N_\mu^T \hat{\lambda}_2 \\
\text{(e)} & 0 = {Z_1'}^T \hat{\lambda}_1, \\
\text{(f)} & 0 = {Z_2'}^T \hat{\lambda}_2.
\end{array}
$$

All quantities are available for all time steps.
An analogous system can be derived for each Gauss-Newton step in the nonlinear case.

Are we finished?

- Optimality conditions (linear and nonlinear) and maximum principle for general DAEs have been derived.
- But we are not there yet.
- To compute the strangeness-free form is difficult in general and impossible for large scale problems. Three $O(n^3)$ nullspace computations per time step. But in many problem (e.g. in the two flow control problems) we can get this for free using the structure.
- The real challenge is the optimality boundary value problem.
- Currently for large scale problems we can only use model reduction (linear case), small scale realization or
- adaptive discretization of Input/Output maps via Adaptive FEM (DWR) Becker/Rannacher 02, Dissertation Schmidt '07, Heiland/M./Schmidt 2009

Discretization of Input/Ouput maps

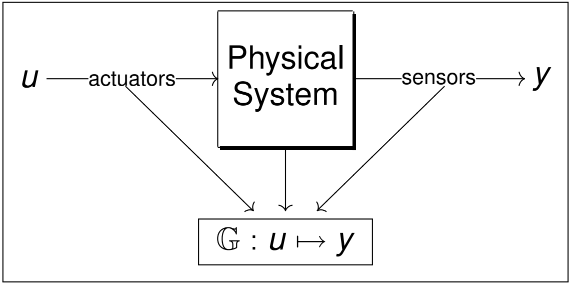


Figure: Schematic illustration of an input/output map, corresponding to a physical system, given e.g. by a set of equations or a numerical solver (black-box approach).

Model red. vs. i./o. map discret.

Current approaches:

- Semi-discretization in space followed by model reduction via proper orthogonal decomposition or balanced truncation.
- High costs for model reduction method.
- But the forward space-time solver is only a tool.
- We are really interested in the behavior of the i./o. map.
- So we should discretize the i./o. map and not the forward problem.
- Theoretical basis, error estimation, adaptive schemes, etc. Dissertation Schmidt '07
- Application to driven cavity flow, Diploma thesis Heiland 2009
- Application to stirred liquid/liquid systems, current dissertation of J. Heiland.

Oseen problem for 2D driven cavity

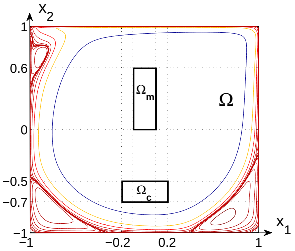


Figure: Schematic illustration of a 2D driven cavity flow and the domains of control and observation, Ω_c and Ω_m, respectively.

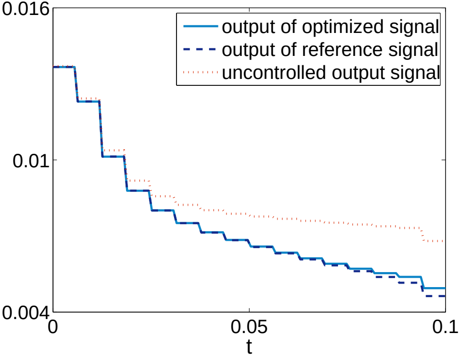
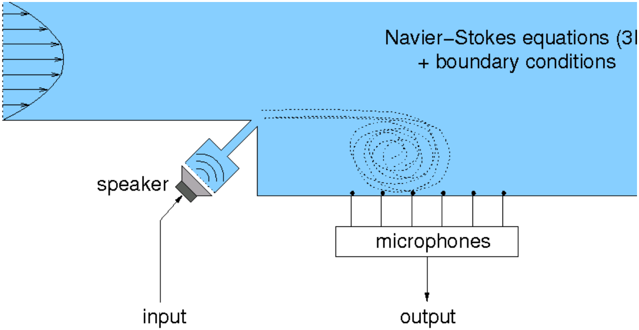
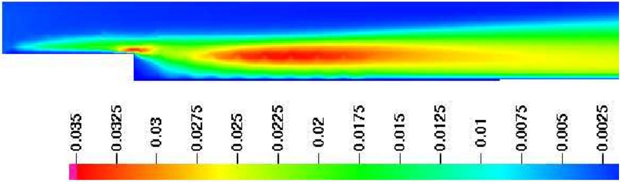


Figure: Illustration of the x_1-component of the output signals

Flow control

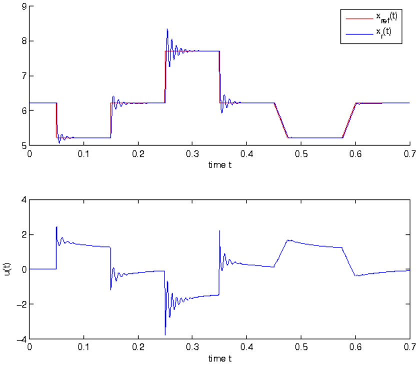


Simulated flow with FEATFLOW



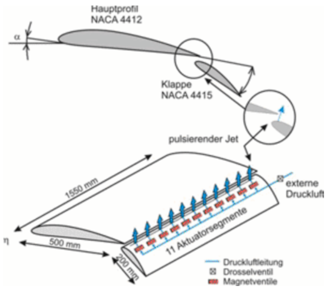
Space discretization leads to an large control system of nonlinear DAEs.

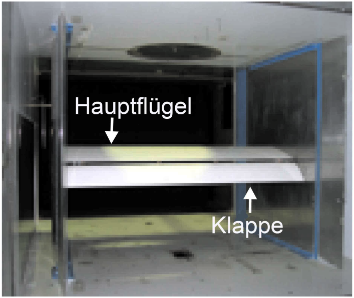
Henning/ Kuzmin/M./Schmidt/Sokolov/Turek ’07. Movement of recirculation bubble following reference curve via controller built into FEATFLOW.



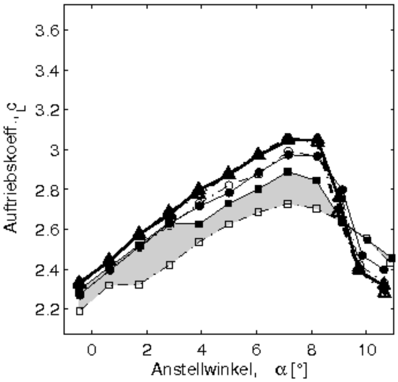
Results obtained with the DFG Collaborative research center SFB 557 TU Berlin.

- Closed loop separation control Becker/King/Petz/Nitsche 07.
- Computational investigation of separation for high lift airfoil flows Schatz/Günther/Thiele '07

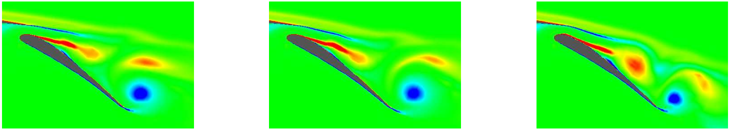

Hauptprofil
NACA 4412
α
Klappe
NACA 4415
pulsierender Jet
1550 mm
externe
Druckluft
η
500 mm
200 mm
11 Aktuatorsegmente
Druckluftleitung
Drosselventil
Magnetventile


Hauptflügel
Klappe

Lift optimization



Flow field for different excitations



Conclusions

- Control problems for general linear and nonlinear DAEs arise in many applications.
- In practice models are often generated automatically.
- Remodelling is necessary.
- Model based feedback control for PDEs is a major challenge.
- We have made some progress in the DAE control theory.
- Without using the structure we cannot solve large scale problems.
- Model reduction or model approximation is essential.
- Methods have been implemented in several applications (not always in a satisfactory way).
- Text book. P. Kunkel and V. Mehrmann, Differential algebraic equations. Analysis and numerical solution. European Mathematical Society, Zürich, 2006.

Thank you very much
for your attention.