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HIERARCHICAL SHAPE OPTIMIZATION :

Cooperation and Competition in Multi-Disciplinary Approaches

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Advanced Methods and perspectives in nonlinear optimization and control
ENSIACET, Toulouse - February 3-5th, 2010

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PDE-Constrained Optimization

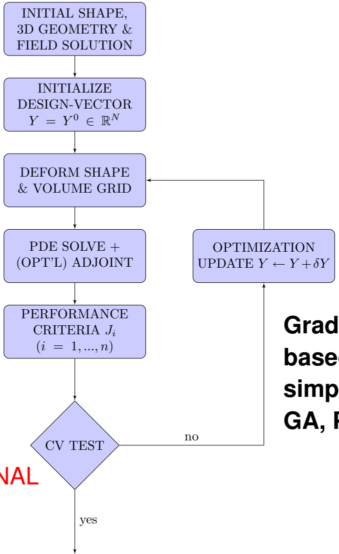
Example of CAD-free Optimum-Shape Design in Aerodynamics

B-spline-type Deformation Parameters

Free-Form Deformation
(Sederberg, Samareh, ...)

Euler/Navier-Stokes

Aerodynamic Coef's
(lift, drag, moments, ...) :
boundary integrals



Gradient-based, simplex, GA, PSO, ...

⟹ CPU-DEMANDING FUNCTIONAL PERFORMANCE EVALUATION

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Hierarchical principles used in numerical shape optimization

Hierarchical Physical Models of High and Low Fidelity

- Simplified Physics
- Statistical Models :
  – state : Proper Orthogonal Decomposition (POD)
  – functional metamodels : surface response, Kriging, ANN, etc

$\longrightarrow$ ANN used in present applications, but not described here

Hierarchical Geometrical Representations

Multilevel algorithms at the stage of analysis (multigrid) or optimization (hierarchical smoothing, one-shot, multilevel parameterization, etc)

$\longrightarrow$ One slide prepared

Hierarchical Treatment of Multi-Disciplinary Optimization

Cooperation and Competition (Nash Games)

$\longrightarrow$ The focus of this talk

Etc

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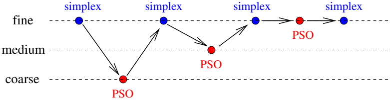
Multilevel shape optimization

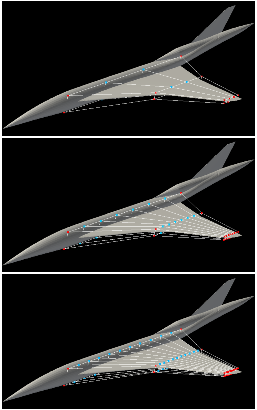
- Basic validation of concept[1]
- Analysis of algebraic model[2]
- Size experiments in compressible aerodynamics[3,4]
- Parameterization self-adaption procedures[5]
- Multilevel shape optimization of antennas[6]
- Stochastic/deterministic Hybridization[7]
- Software: FAMOSA platform + Scilab toolbox
- Participation in two European short courses on optimization (ERCOFTAC, Von Karman Institute)
- Invited conference at the German Aerospace Lab (DLR Braunschweig)
- On-going: extension to *algebraic* hierarchical basis

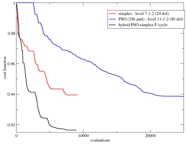
1 J. Computational Physics, 2007
2 Advances in Numerical Mathematics, 2006
3 B. Abou El Majd's Doctoral Thesis, 2007
4 European J. of Computational Mechanics, 2008
5 European Series in Applied and Industrial Mathematics, 2007
6 B. Chaigne's Doctoral Thesis, 2009
7 *Optimisation Multidisplinaire en Mécanique*, Hermès, 2009

stiff + multimodal problem $\Longrightarrow$ multilevel + hybrid + parallel algorithm



Free-Form Deformation





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Multi-objective optimization

Examples in aerodynamic design in Aeronautics

- Criteria are usually **field functionals**, thus costly-to-evaluate
  - **Multi-criterion** *(single-flow conditions)*

    – e.g. lift and moments (stability/maneuverability)
  - **Multi-point** *(several flow conditions)* **e.g.:**

    – drag reduction at several cruise conditions (towards "robust design"), or

    – lift maximization at take-off or landing conditions, drag reduction at cruise
  - **Multi-discipline** *(Aerodynamics + others)*

    – e.g. aerodynamic performance versus criteria related to: structural design, acoustics, thermal loads, etc

    – Special case: **'preponderant'** or **'fragile'** discipline
- **Objective:** *devise cost-efficient algorithms to determine appropriate trade-offs between concurrent minimization problems associated with the criteria* J_A, J_B, *...*

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Notion of dominance/non-dominance

for minimization problems

Let $Y \in \mathbb{R}^N$ denote the vector of design variables.

If *several minimization problems* are to be considered *concurrently*, a design point Y^1 is said to *dominate in efficiency* the design point Y^2, symbolically

$$Y^1 \succ Y^2$$

iff, for all the criteria to be minimized $J = J_A, J_B, \ldots$

$$J\left(Y^1\right) \leq J\left(Y^2\right)$$

and at least one of these inequalities is strict.

Otherwise: non-dominance $\iff Y^1 \not\succ Y^2$ and $Y^2 \not\succ Y^1$

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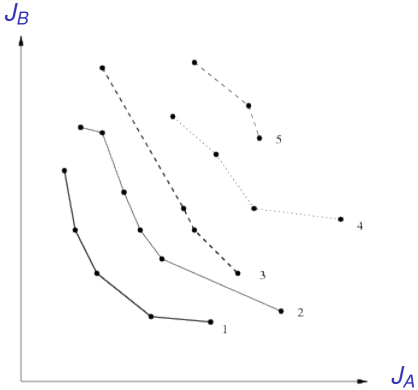
Pareto fronts

GA's relying on fitness function related to front index

- NPGA : Niched Pareto Genetic Algorithm, Goldberg *et al*, 1994
- NSGA : Nondominated Sorting Genetic Algorithm, Srinivas & Deb, 1994
- MOGA : Multiobjective Genetic Algorithm, Fonseca *et al*, 1998
- SPEA : Strength Pareto Evolutionary Algorithm, Zitzler *et al*, 1999



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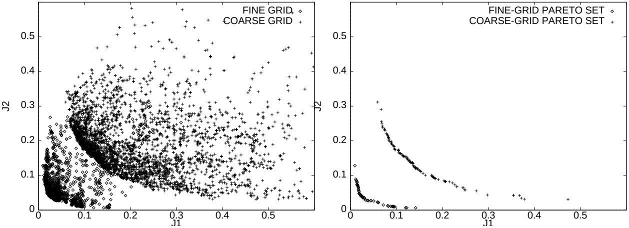
Example of airfoil shape concurrent optimization

J_A: transonic- cruise pressure drag (minimization);
J_B: subsonic take-off or landing lift (maximization);
Euler equations; Marco *et al*, INRIA RR 3686 (1999).



Accumulated populations and Pareto sets
(independent simulations on a coarse and a fine meshes)
https://hal.inria.fr/inria-00072983

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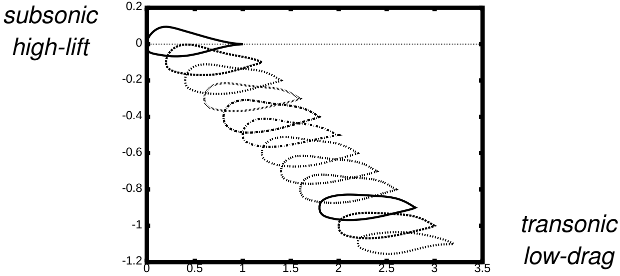
Airfoil shapes of Pareto-equilibrium front

Non-dominated designs



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Numerical efficiency

- Principal merits
  - Very rich unbiased information provided to designer
  - Very general : applies to non-convex, or discontinuous Pareto-equilibrium fronts
- Main disadvantages
  - Incomplete sorting (decision still to be made)
  - Very costly

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Alternatives to costly Pareto-front identification

1. Agglomerated criterion

Minimize agglomerated criterion

$$J = \alpha J_A + \beta J_B + \dots$$

for some appropriate constants α, β, ...
$[\alpha] \sim [J_A]^{-1}, \quad [\beta] \sim [J_B]^{-1}$

Unphysical, arbitrary, lacks of generality, ...

Similar alternative :

- First, solve n independent single-objective minimizations :

$$J^* = \min J \qquad \text{for } J = J_A,\ J_B,\ \dots$$

- Second, solve the following multi-constrained single-objective minimization problem :

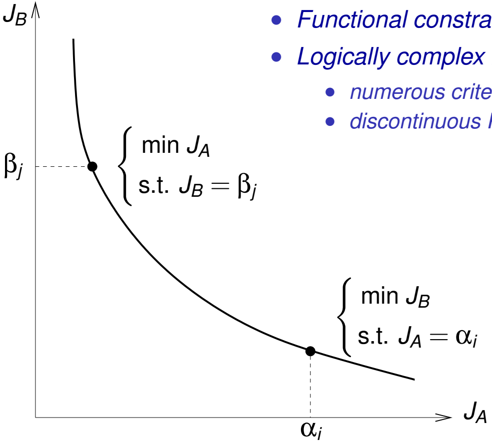
$$\min\ T \qquad \text{subject to : } J_A \leq J_A^* + \alpha T,\ J_B \leq J_B^* + \beta T,\ \dots$$

Alternatives (cont'd)

2. Pointwise determination of Pareto front

Shortcomings:

- *Functional constraints*
- *Logically complex in case of:*
  - *numerous criteria*
  - *discontinuous Pareto front*



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Alternatives (cont'd)

3. Multi-level modeling, METAMODELS

- For each discipline A, B, ..., consider a hierarchy of models and corresponding criteria based on a METAMODEL (*POD, ANN, Kriging, surface response, interpolation, ...*);
- Devise a multi-level strategy for multi-objective optimization in which complexity is gradually introduced.

This is the strategy adopted in the *« OMD » Network on Multi-Disciplinary Optimization* supported by the French ANR.

See also: web site of Prof. K. Giannakoglou for acceleration techniques using parallel computing:
http://velos0.ltt.mech.ntua.gr/research/

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Alternatives (end)

4. Game strategies

- Symmetrical game:
  Nash
- Unsymmetrical or hierarchical game:
  Stackelberg (leader-follower)

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Nash games involving primitive variables

Prototype example of equilibrium between two criteria

- Split the design vector Y into two sub-vectors:

$$Y = (Y_A, Y_B)$$

and use them as the *strategies* of two independent *players A* and *B* in charge of minimizing the criteria J_A and J_B respectively.

- Seek an equilibrium point $\overline{Y} = \left(\overline{Y}_A, \overline{Y}_B\right)$ such that:

$$\overline{Y}_A = \text{Argmin}_{Y_A} J_A \left(Y_A, \overline{Y}_B\right)$$

and

$$\overline{Y}_B = \text{Argmin}_{Y_B} J_B \left(\overline{Y}_A, Y_B\right)$$

... many examples in market or social negociations.

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Possible parallel algorithm implementation

Often requires under-relaxation to converge

1. Initialize both sub-vectors:

$$Y_A := Y_A^{(0)} \qquad Y_B := Y_B^{(0)}$$

2. Perform in parallel:

- **Retrieve** and maintain fixed $Y_B = Y_B^{(0)}$
- **Update** Y_A **alone**

by K_A design cycles to minimize or reduce $J_A\left(Y_A, Y_B^{(0)}\right)$; obtain $Y_A^{(K_A)}$.

//

- **Retrieve** and maintain fixed $Y_A = Y_A^{(0)}$
- **Update** Y_B **alone**

by K_B design cycles to minimize or reduce $J_B\left(Y_A^{(0)}, Y_B\right)$; obtain $Y_B^{(K_B)}$.

3. Update sub-vectors to prepare information exchange

$$Y_A^{(0)} := Y_A^{(K_A)} \qquad Y_B^{(0)} := Y_B^{(K_B)}$$

and return to step 2 or stop (if convergence achieved).

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Invariance of Nash equilibrium

through arbitrary scaling laws

Let Φ and Ψ be smooth, strictly monotone-increasing functions.

The Nash equilibrium point $\left(\overline{Y}_A, \overline{Y}_B\right)$ associated with the formulation:

$$\overline{Y}_A = \mathrm{Argmin}_{Y_A}\, \Phi\left[J_A\left(Y_A, \overline{Y}_B\right)\right]$$

and

$$\overline{Y}_B = \mathrm{Argmin}_{Y_B}\, \Psi\left[J_B\left(\overline{Y}_A, Y_B\right)\right]$$

does not depend on Φ or Ψ.

The **split of territories,** $Y = (Y_A, Y_B)$, is therefore the **sole critical element** in a Nash game.

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My basic problematics

Given smooth criteria $J_A(Y)$, $J_B(Y)$, ... ($Y \in \mathbb{R}^N$) and exact or approximate information on gradients and Hessians, determine an appropriate split of design variables Y to realize a multi-criterion optimization via a sensible Nash game.

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Example of equilibrium with physically-relevant split

From *Tang-Désidéri-Périaux, J. Optimization Theory and Applications (JOTA, Vol. 135, No. 1, October 2007)*

Shape parameterization :

Hicks-Henne basis functions

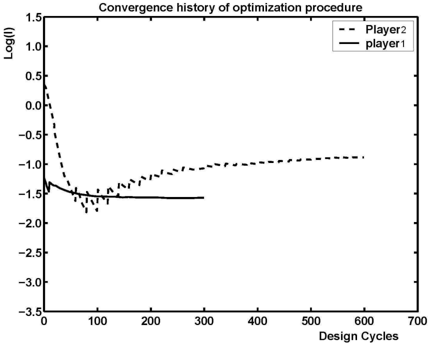
Lift-Control (C_L) in Subsonic conditions (1st design point)

Drag-Control (C_D) in Transonic conditions (2nd design point)

$$\min_{\Gamma_1} J_A = \int_{\Gamma_c} (p - p_{\text{sub}})^2 \qquad \min_{\Gamma_2} J_B = \int_{\Gamma_c} (p - p_{\text{trans}})^2$$

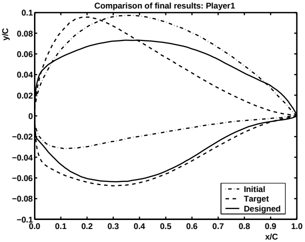
Exchange of information every 5 +10 parallel design iterations

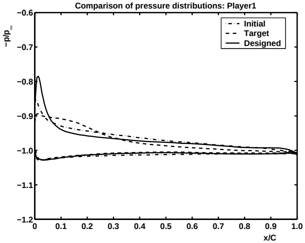
Convergence of the two criteria towards the Nash equilibrium



Shapes and pressure distribution at 1st design point

Subsonic flow





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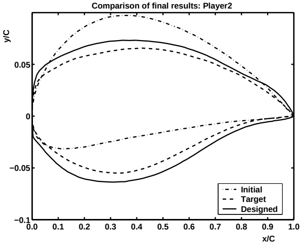
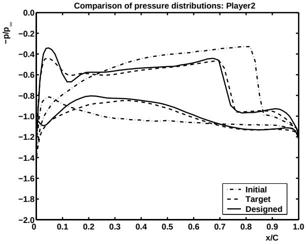
Shapes and pressure distribution at 2nd design point

Transonic flow





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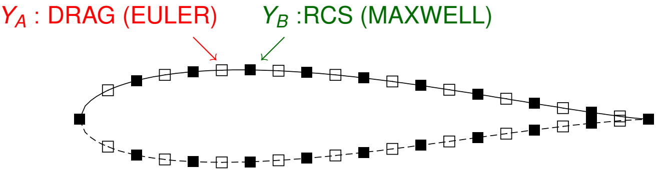
Another type of territory split

for multi-disciplinary optimization; from *H.Q. Chen-Périaux-Désidéri*



Two players A and B, controling Y_A (■) and Y_B (□) respectively, optimize their own criterion J_A (e.g. DRAG) or J_B (e.g. RCS), and exchange information at regular intervals.

Geometrical regularity is maintained.

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Computational efficiency

- Principal merits
  - Also fairly general (no penalty constants to choose)
  - Applicable to optimization algorithms of all types (deterministic/evolutionary) and their combinations
  - Much more economical
- Shortcomings
  - Relation to Pareto-equilibrium front seldomly clear
  - **Defining territories pertinently raises fundamental questions**

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Hierarchical Shape Optimization

Jean-Antoine Désidéri

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A difficult two-discipline wing shape optimization

Jeux dynamiques en optimisation couplée fluide-structure. In: Abou El Majd, Doctoral Thesis, University of Nice-Sophia Antipolis, September 2007.

$$Y = (Y_A, Y_S) \in \mathbb{R}^N$$

- **Aerodynamics –** $\min_{Y_A} J_A$:

$$J_A = \frac{C_D}{C_{D_0}} + 10^4 \max\left(0, 1 - \frac{C_L}{C_{L_0}}\right)$$

- **Structural design –** $\min_{Y_S} J_S$:

$$J_S = \iint_S \|\sigma.n\| \, dS + K_1 \max\left(0, 1 - \frac{V}{V_A}\right) + K_2 \max\left(0, \frac{S}{S_A} - 1\right)$$

stress σ calculated by EDF code *ASTER;* S_A and V_A wing surface and volume after aerodynamic optimization

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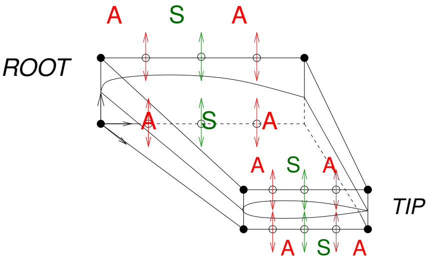
A trial splitting strategy using primitive variables

A total of 12 degrees of freedom $(4 \times 1 \times 1)$

Alternating split of root and tip parameters

Structural territory:
4 vertical displacements of mid-control-points of upper and lower surfaces, $Y_S \in \mathbb{R}^4$

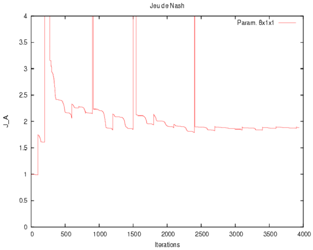
Aerodynamic territory: 8 remaining vertical displacements, $Y_A \in \mathbb{R}^8$



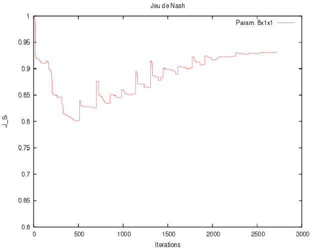
Convergence of the two criteria (simplex iterations)

Asymptotic Nash equilibrium

PRESSURE DRAG (J_A)



STRESS INTEGRAL (J_S)



Very antagonistic coupling

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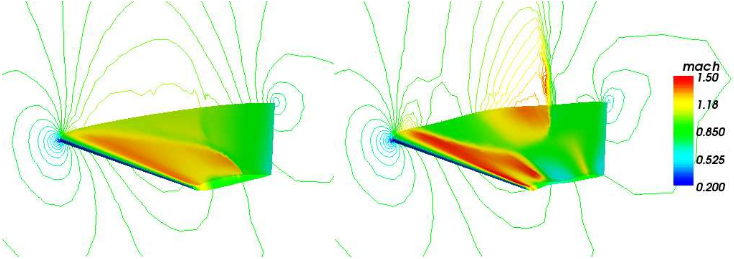
Aerodynamic optimum shape and shape resulting from inappropriate Nash equilibrium



Aerodynamics optimized alone   Unacceptable coupling

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Recommended Eigensplitting

Split of Territories in Concurrent Optimization, J.A.D.,
INRIA Research Report 6108, 2007;
https://hal.inria.fr/inria-00127194

(1) First Phase : optimize primary discipline (A) alone

$$\min_{Y \in \mathbb{R}^N} J_A(Y)$$

subject to K equality constraints:

$$g(Y) = (g_1, g_2, \dots, g_K)^T = 0$$

Get :

1. Single-discipline optimal design vector : Y_A^*
2. Hessian matrix (primary discipline) : $H_A^* = H_A(Y_A^*)$
3. Active constraint gradients : $\nabla g_k^* = \nabla g_k(Y_A^*)$ $(k = 1, 2, \dots, K)$

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Eigensplitting - cont'd

(2) Construct orthogonal basis in preparation of split

1. Transform $\{\nabla g_k^*\}$ into $\{\omega^k\}$ $(k = 1, 2, ..., K)$ by Gram-Schmidt orthogonalization process, and form the projection matrix :

$$P = I - \left[\omega^1\right]\left[\omega^1\right]^t - \left[\omega^2\right]\left[\omega^2\right]^t - \cdots - \left[\omega^K\right]\left[\omega^K\right]^t$$

2. Restrict Hessian matrix to subspace tangent to constraint surfaces :

$$\boxed{H'_A = P\,H_A^*\,P}$$

3. Diagonnalize matrix H'_A,

$$H'_A = \Omega\,\text{Diag}(h'_k)\,\Omega^t$$

using an appropriate ordering of the eigendirections :

$$h'_1 = h'_2 = ... = h'_K = 0\,;\ h'_{K+1} \geq h'_{K+2} \geq ... \geq h'_N$$

Tail column-vectors of matrix Ω correspond to directions of least sensitivity of primary criterion J_A subject to constraints.

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Eigensplitting - end

(3) Organize the Nash game in the eigenvector-basis Ω

Consider the splitting of parameters defined by:

$$Y = Y_A^* + \Omega \begin{pmatrix} U \\ V \end{pmatrix}, \; U = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-p} \end{pmatrix}, \; V = \begin{pmatrix} v_p \\ \vdots \\ v_1 \end{pmatrix} \tag{1}$$

Let ε be a small positive parameter ($0 \le \varepsilon \le 1$), and let $\overline{Y}_\varepsilon$ denote the Nash equilibrium point associated with the concurrent optimization problem:

$$\begin{cases} \min\limits_{U \in \mathbb{R}^{N-p}} J_A \\ \text{Subject to: } g = 0 \end{cases} \quad \text{and} \quad \begin{cases} \min\limits_{V \in \mathbb{R}^{p}} J_{AB} \\ \text{Subject to: } \textit{no constraints} \end{cases} \tag{2}$$

in which again the constraint $g = 0$ is not considered when $K = 0$, and

$$J_{AB} := \frac{J_A}{J_A^*} + \varepsilon \left(\theta \, \frac{J_B}{J_B^*} - \frac{J_A}{J_A^*} \right) \tag{3}$$

where θ is a strictly-positive relaxation parameter ($\theta < 1$: under-relaxation; $\theta > 1$: over-relaxation).

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Theorem; setting 1.

Split of Territories in Concurrent Optimization, J.A.D., INRIA Research Report 6108, 2007; https://hal.inria.fr/inria-00127194

Let N, p and K be positive integers such that:

$$1 \leq p < N, \quad 0 \leq K < N-p \tag{4}$$

Let J_A, J_B and, if $K \geq 1$, $\{g_k\}$ $(1 \leq k \leq K)$ be $K+2$ smooth real-valued functions of the vector $Y \in \mathbb{R}^N$. Assume that J_A and J_B are positive, and consider the following primary optimization problem,

$$\min_{Y \in \mathbb{R}^N} J_A(Y) \tag{5}$$

that is either unconstrained ($K = 0$), or subject to the following K equality constraints:

$$g(Y) = (g_1, g_2, ..., g_K)^T = 0 \tag{6}$$

Assume that the above minimization problem admits a local or global solution at a point $Y_A^* \in \mathbb{R}^N$ at which $J_A^* = J_A(Y_A^*) > 0$ and $J_B^* = J_B(Y_A^*) > 0$, and let H_A^* denote the Hessian matrix of the criterion J_A at $Y = Y_A^*$.

If $K = 0$, let $P = I$ and $H_A' = H_A^*$; otherwise, assume that the constraint gradients, $\{\nabla g_k^*\}$ $(1 \leq k \leq K)$, are linearly independent.

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Theorem; setting 2.

Apply the Gram-Schmidt orthogonalization process to the constraint gradients, and let $\{\omega^k\}$ $(1 \leq k \leq K)$ be the resulting orthonormal vectors. Let P be the matrix associated with the projection operator onto the K-dimensional subspace tangent to the hyper-surfaces $g_k = 0$ $(1 \leq k \leq K)$ at $Y = Y_A^*$,

$$P = I - [\omega^1]\,[\omega^1]^t - [\omega^2]\,[\omega^2]^t - \cdots - [\omega^K]\,[\omega^K]^t \tag{7}$$

Consider the following real-symmetric matrix:

$$H'_A = P\,H_A^*\,P \tag{8}$$

Let Ω be an orthogonal matrix whose column-vectors are normalized eigenvectors of the matrix H'_A organized in such a way that the first K are precisely $\{\omega^k\}$ $(1 \leq k \leq K)$, and the subsequent $N - K$ are arranged by decreasing order of the eigenvalue

$$h'_k = \omega^k \,.\, H'_A\,\omega^k = \omega^k \,.\, H_A^*\,\omega^k \quad (K+1 \leq k \leq N) \tag{9}$$

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Theorem; setting 3.

Consider the splitting of parameters defined by:

$$Y = Y_A^* + \Omega \begin{pmatrix} U \\ V \end{pmatrix},\ U = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-p} \end{pmatrix},\ V = \begin{pmatrix} v_p \\ \vdots \\ v_1 \end{pmatrix} \tag{10}$$

Let ε be a small positive parameter ($0 \le \varepsilon \le 1$), and let $\overline{Y}_\varepsilon$ denote the Nash equilibrium point associated with the concurrent optimization problem:

$$\begin{cases} \min\limits_{U\in\mathbb{R}^{N-p}} J_A \\ \text{Subject to: } g = 0 \end{cases} \quad \text{and} \quad \begin{cases} \min\limits_{V\in\mathbb{R}^{p}} J_{AB} \\ \text{Subject to: } \textit{no constraints} \end{cases} \tag{11}$$

in which again the constraint $g = 0$ is not considered when $K = 0$, and

$$J_{AB} := \frac{J_A}{J_A^*} + \varepsilon \left(\theta \frac{J_B}{J_B^*} - \frac{J_A}{J_A^*} \right) \tag{12}$$

where θ is a strictly-positive relaxation parameter ($\theta < 1$: under-relaxation; $\theta > 1$: over-relaxation).

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Theorem; conclusions 1.

Then:

- *[Optimality of orthogonal decomposition]* If the matrix H'_A is positive semi-definite, which is the case in particular if the primary problem is unconstrained ($K = 0$), or if it is subject to linear equality constraints, its eigenvalues have the following structure:

$$h'_1 = h'_2 = \dots = h'_K = 0 \quad h'_{K+1} \geq h'_{K+2} \geq \dots \geq h'_N \geq 0 \tag{13}$$

and the tail associated eigenvectors $\{\omega^k\}$ ($K+1 \leq k \leq N$) have the following variational characterization:

$$\begin{aligned}
\omega^N &= \text{Argmin}_\omega |\omega . H_A^* \omega| \quad \text{s.t. } \|\omega\| = 1 \text{ and } \omega \perp \{\omega^1, \omega^2, \dots, \omega^K\} \\
\omega^{N-1} &= \text{Argmin}_\omega |\omega . H_A^* \omega| \quad \text{s.t. } \|\omega\| = 1 \text{ and } \omega \perp \{\omega^1, \omega^2, \dots, \omega^K, \omega^N\} \\
\omega^{N-2} &= \text{Argmin}_\omega |\omega . H_A^* \omega| \quad \text{s.t. } \|\omega\| = 1 \text{ and } \omega \perp \{\omega^1, \omega^2, \dots, \omega^K, \omega^N, \omega^{N-1}\} \\
&\vdots
\end{aligned} \tag{14}$$

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Theorem; conclusions 2 (cont'd).

- *[Preservation of optimum point as a Nash equilibrium]* For $\varepsilon = 0$, a Nash equilibrium point exists and it is:

$$\overline{Y}_0 = Y_A^* \qquad (15)$$

- *[Robustness of original design]* If the Nash equilibrium point exists for $\varepsilon > 0$ and sufficiently small, and if it depends smoothly on this parameter, the functions:

$$j_A(\varepsilon) = J_A\left(\overline{Y}_\varepsilon\right), \quad j_{AB}(\varepsilon) = J_{AB}\left(\overline{Y}_\varepsilon\right) \qquad (16)$$

are such that:

$$j_A'(0) = 0 \qquad (17)$$

$$j_{AB}'(0) = \theta - 1 \leq 0 \qquad (18)$$

and

$$j_A(\varepsilon) = J_A^* + O(\varepsilon^2) \qquad (19)$$

$$j_{AB}(\varepsilon) = 1 + (\theta - 1)\varepsilon + O(\varepsilon^2) \qquad (20)$$

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Theorem; conclusions 3 (end).

- In case of linear equality constraints, the Nash equilibrium point satisfies identically:

$$u_k(\varepsilon) = 0 \quad (1 \leq k \leq K) \tag{21}$$

$$\overline{Y}_\varepsilon = Y_A^* + \sum_{k=K+1}^{N-p} u_k(\varepsilon)\,\omega^k + \sum_{j=1}^{p} v_j(\varepsilon)\,\omega^{N+1-j} \tag{22}$$

- For $K = 1$ and $p = N - 1$, the Nash equilibrium point $\overline{Y}_\varepsilon$ *is Pareto optimal*.

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Proof; (1)

- Optimality of initial point (Y_A^*):

$$\nabla J_A^* + \sum_{k=1}^{K} \lambda_k \nabla g_k^* = 0\,, \quad g = 0$$

$$\Longrightarrow \nabla J_A^* \in Sp\left(\omega^1\,, \omega^2\,, \ldots, \omega^K\right) \text{ (Gram-Schmidt)}$$

- For $\varepsilon = 0$:

$$J_A = J\,, \quad J_{AB} = \frac{J_A}{J_A^*} = \text{const.} \times J\,, \quad \nabla J_{AB} = \frac{J_A}{J_A^*} = \text{const.} \times \nabla J$$

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Proof; (2)

- Optimality of sub-vector U w.r.t. criterion $J_A = J$ for fixed V and under equality constraints:

$$\left(\frac{\partial J}{\partial U}\right)_V = \nabla J \, . \left(\frac{\partial Y}{\partial U}\right)_V = -\sum_{k=0}^{K} \lambda_k \nabla g_k^* \, . \left(\frac{\partial Y}{\partial U}\right)_V = -\sum_{k=0}^{K} \lambda_k \left(\frac{\partial g_k^*}{\partial U}\right)_V$$

$$\Longrightarrow \left(\frac{\partial}{\partial U}\right)_V \left(J + \sum_{k=0}^{K} \lambda_k \, g_k\right) = 0 \text{ and } g = 0$$

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Proof; (3)

- Optimality of sub-vector V w.r.t. criterion $J_{AB} \sim J$ for fixed U:

$$Y = Y_A^* + \Omega \begin{pmatrix} U \\ V \end{pmatrix}$$

$$\left(\frac{\partial J}{\partial V}\right)_U = \nabla J \,.\, \left(\frac{\partial Y}{\partial V}\right)_U = \nabla J \,.\, \underbrace{\Omega \begin{pmatrix} 0 & 0 \\ 0 & I_p \end{pmatrix}}_{} = 0$$

$$\in Sp\left(\omega^{N-p+1}, \dots, \omega^{N}\right)$$

provided $K < N - p + 1$.

$\Longrightarrow$ $Y_A^* = \overline{Y}_0$ (initial Nash equilibrium point)

$\Longrightarrow$ Continuum of equilibrium points parameterized by ε □

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Proof; (4)

Case of linear equality constraints

- Linearly-independent constraint gradient vectors $\{ L_k = \nabla g_k^* \}$ $(1 \leq k \leq K)$ (otherwise reduce K):

$$g_k = L_k \,.\, Y - b_k = L_k \,.\, (Y - Y_A^*) = 0 \quad (1 \leq k \leq K)$$

- Continuum of Nash equilibrium points parameterized by ε:

$$\overline{Y}_\varepsilon = Y_A^* + \sum_{j=1}^{N-p} u_j(\varepsilon)\,\omega^j + \sum_{j=1}^{p} v_j(\varepsilon)\,\omega^{N+1-j}$$

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Proof; (5)

Case of linear equality constraints (end)

- By orthogonality of the eigenvectors, and since $L_k = \nabla g_k^* \in Sp\left(\omega^1, \ldots, \omega^K\right)$, the equality constraints,

$$< L_k, \sum_{j=1}^{N-p} u_j(\varepsilon)\,\omega^j + \sum_{j=1}^{p} v_j(\varepsilon)\,\omega^{N+1-j} > = 0 \quad (1 \le k \le K)$$

simplify to:

$$< L_k, \sum_{j=1}^{K} u_j(\varepsilon)\,\omega^j > = 0 \quad (1 \le k \le K)$$

and this is an invertible homogeneous linear system of K equations for the K unknowns $\{\, u_j(\varepsilon)\,\}$ $(1 \le j \le K)$.

$$\Longrightarrow u_1(\varepsilon) = u_2(\varepsilon) = \ldots = u_K(\varepsilon) = 0\,,\ \overline{Y}_\varepsilon - Y_A^* \perp \nabla J_A^*,\ j_A'(0) = 0 \quad \square$$

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Proof; (6)

Case of nonlinear equality constraints

- Define neighboring Nash equilibrium point associated with linearized constraints, $\overline{Y}_\varepsilon^L$, for which:

$$J_A\left(\overline{Y}_\varepsilon^L\right) = J_A^* + O(\varepsilon^2)$$

- Define projections:

$$\overline{Y}_\varepsilon - \overline{Y}_\varepsilon^L = v + w$$

where $v \in Sp\,(\,L_1, L_2, ..., L_K\,)$ and $w \in Sp\,(\,L_1, L_2, ..., L_K\,)^\perp$.

- Assume local regularity and smoothness of the hyper-surfaces $g_k = 0$:

$$v = O(\varepsilon)\,, \quad w = O(\varepsilon^2)$$

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Proof; (7)

Case of nonlinear equality constraints (end)

- Then:

$$
\begin{aligned}
j_A(\varepsilon) &= J_A\left(\overline{Y}_\varepsilon\right) \\
&= J_A\left(\overline{Y}^L_\varepsilon + v + w\right) \\
&= J_A\left(\overline{Y}^L_\varepsilon\right) + \nabla J_A\left(\overline{Y}^L_\varepsilon\right).(v+w) + O(\varepsilon^2) \\
&= J_A\left(\overline{Y}^L_\varepsilon\right) + \nabla J_A^*.(v+w) + O(\varepsilon^2) \quad \text{provided } \nabla J_A^* \text{ is smooth} \\
&= J_A\left(\overline{Y}^L_\varepsilon\right) + O(\varepsilon^2) \quad \text{since } \nabla J_A^*.v = 0 \text{ and } \nabla J_A^*.w = O(\varepsilon^2) \\
&= J_A^* + O(\varepsilon^2) \quad \text{and } j_A'(0) = 0 \text{ again.}
\end{aligned}
$$

$\Longrightarrow$ Concerning the primary criterion J_A, the initial design is robust w.r.t. small perturbations in ε □

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Proof; (8) (end)

- Lastly, the secondary criterion satisfies:

$$j_{AB}(\varepsilon) = \frac{j_A(\varepsilon)}{J_A^*} + \varepsilon \left(\theta \frac{j_B(\varepsilon)}{J_B^*} - \frac{j_A(\varepsilon)}{J_A^*} \right)$$

$$j'_{AB}(0) = 0 + 1 \times (\theta - 1) + 0 = \theta - 1 \leq 0 \quad \square$$

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Example

Variables:

$$Y = \left(y_0, y_1, y_2, y_3\right) \in \mathbb{R}^4$$

Primary problem:

$$\min J_A(Y) = \sum_{k=0}^{3} \frac{y_k^2}{A^k}$$

Subject to: $g = 0$

Secondary problem:

$$\min J_B(Y) = \sum_{k=0}^{3} y_k^2$$

Subject to: *no constraints*

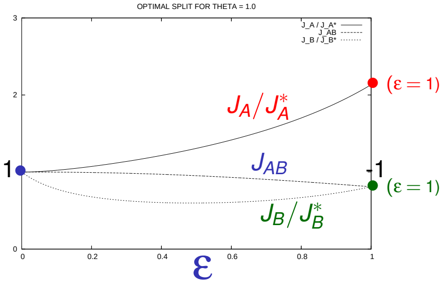
A: antagonism parameter ($A \geq 1$)

$$g = \sum_{k=0}^{3} \left(y_k - A^k\right), \text{ or } y_0^4\, y_1^3\, y_2^2\, y_3 - 96\sqrt{3} = 0$$

Case of a nonlinear constraint :

$$g = y_0^4\, y_1^3\, y_2^2\, y_3 - 96\sqrt{3} = 0$$

Continuation method ($A = 3$, $\theta = 1$)



The continuum of Nash equilibriums as ε varies

NOTE: the function $j_B(\varepsilon) = \frac{J_B(\overline{Y}_\varepsilon)}{J_B^*}$ is not monotone ! ($\varepsilon^* \sim 0.487$)

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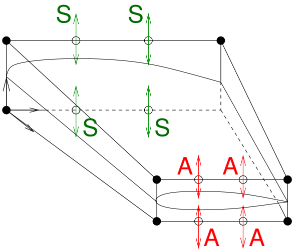
Aerodynamic & structural concurrent optimization exercise

From B. Abou El Majd's Doctoral Thesis

First strategy: split of primitive variables (after many unsuccessful trials)

A total of 8 degrees of freedom $(3 \times 1 \times 1)$

Structural criterion J_S: root, $Y_S \in \mathbb{R}^4$



Aerodynamic criterion J_A: tip, $Y_A \in \mathbb{R}^4$

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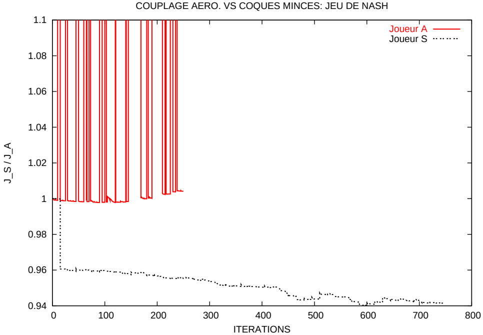
Aerodynamic metamodel vs structural model

Split of primitive variables - convergence of the two criteria



- **Nash equilibrium not completely reached (yet)**
- **But acceptable improved solution attained**

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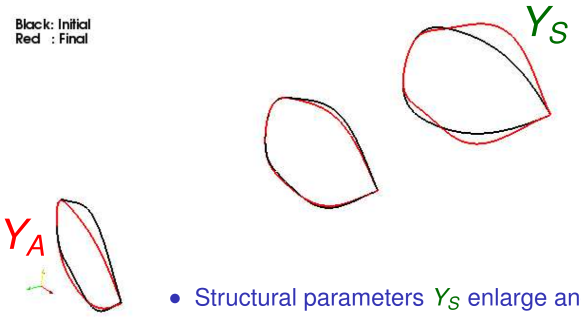
Split of primitive variables - evolution of cross sections



- Structural parameters Y_S enlarge and round out root; shape altered in shock region
- Aerodynamic parameters Y_A attempt to compensate in the critical tip region

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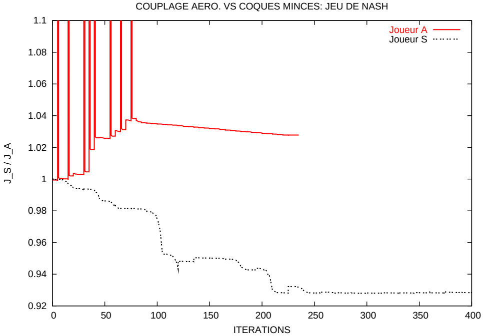
Projected-Hessian-based Eigensplit - convergence of the two criteria



- **More stable Nash equilibrium reached**
- Aero. criterion: $<3\%$ degradation; Structural: $\sim 7\%$ gain

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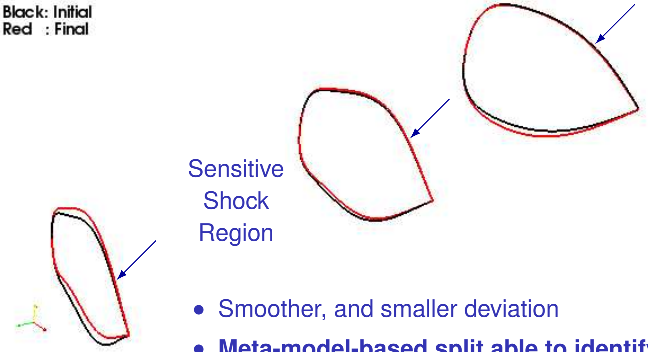

- Smoother, and smaller deviation
- **Meta-model-based split able to identify structural parameters preserving the geometry spanwise in the shock region !!!**

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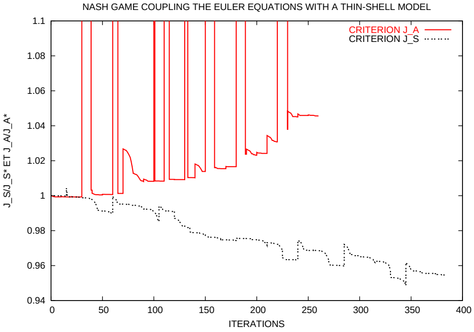
Split of primitive variables - convergence of the two criteria



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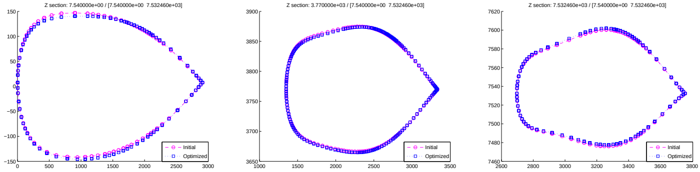
Split of primitive variables - evolution of cross sections

a) Root | b) Mid-span | c) Tip



ONLY MINUTE SHAPE VARIATIONS PERMITTED BY CONSTRAINTS $\Longrightarrow$ poor performance of optimization

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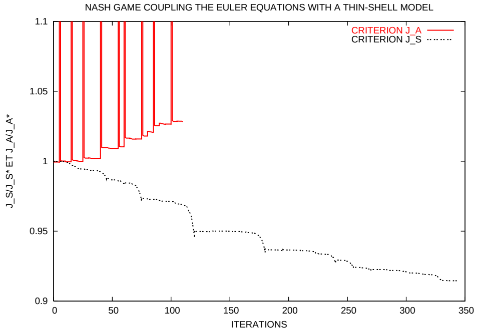
Projected-Hessian-based Eigensplit - convergence of the two criteria



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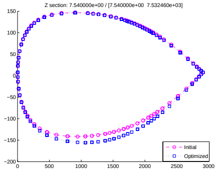
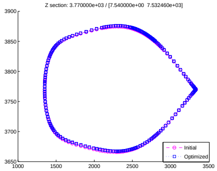
Projected-Hessian-based Eigensplit - evolution of cross sections

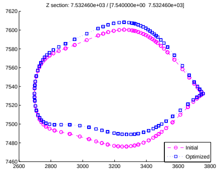
a) Root



b) Mid-span



c) Tip



A SUBSPACE RESPECTING CONSTRAINTS HAS BEEN FOUND IN WHICH OPTIMIZATION CAN PERFORM

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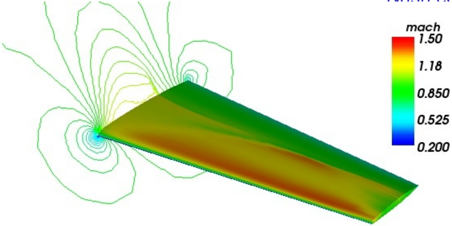
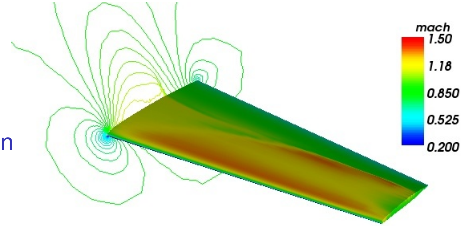
Mach number surface distributions

Original aerodynamic absolute optimum vs recommended Nash equilibrium solution



Original aerodynamic absolute optimum

Aero-structural Nash equilibrium solution



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Summary (1)

- *An abstract split of territories* is recommended for cases in which the design must remain sub-optimal w.r.t. a given *primary, i.e. preponderant or fragile functional*. The split is defined through an eigenproblem involving the Hessian matrix and the constraint gradient vectors. These quantities may be approximated through *meta-models*.
- *A continuum of Nash equilibriums originating from the point Y_A^** of optimality of the primary functional alone (subject to constraints), can be identified through a *perturbation formulation.* The property of *preservation of the initial optimum* ($\overline{Y}_0 = Y_A^*$), is more trivially satisfied for unconstrained problems ($\nabla J_A^* = 0$).

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Summary (end)

- *Robustness:* along the continuum, *small deviations* away from the initial point $\overline{Y}_0 = Y_A^*$ *induce second-order variations* in the primary functional: $J_A\left(\overline{Y}_\varepsilon\right) = J_A^* + O\left(\varepsilon^2\right)$; *$J_A$ is 'insensitive' to small ε.*
- *Aerodynamic-Structural coupled shape optimization exercise:*
  - the *ANN-based automatic eigen-splitting* was found able to recognize that the *structural parameters should not alter the shock region;*
  - as a result, *a gain of about 8 % in the structural criterion* has been achieved, at the expense of only a *3 % degradation in the aerodynamic criterion.*

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Initial setting

Initial design vector :

$Y^0 \in \mathcal{H}$ (usually $\mathcal{H} = \mathbb{R}^N$; $N \geq n$)

Smooth criteria :

$J_i(Y) \quad (1 \leq i \leq n)$ (at least C^2)

Available gradients : $u_i^0 = \nabla J_i^0$
Hessian matrices : H_i^0, and their norms, e.g. :

$$\left\| H_i^0 \right\| = \sqrt{\text{trace}\left[\left(H_i^0\right)^2\right]}$$

Superscript ^{0} indicates an evaluation at $Y = Y^0$

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Preliminary transformation of criteria

J_i is replaced by:

$$\tilde{\tilde{J}}_i\,(Y) = \exp\left(\alpha_i \frac{\|H_i^0\|}{\|\nabla J_i^0\|^2}\,(J_i - J_i^0)\right) + \varepsilon_0\,\phi\left(\frac{\|Y - Y^0\|^2}{R^2} - 1\right)$$

$$\phi(x) = 0 \text{ if } x \leq 0\,, \text{ and } x \exp\left(-\frac{1}{x^2}\right) \text{ if } x > 0 \quad (\text{of class } C^\infty)$$

$$\text{Scaling : } \alpha_i \frac{\|H_i^0\|}{\|\nabla J_i^0\|} = \frac{\gamma}{R} \sim 1$$

$$\mathcal{B}_R = \mathcal{B}\left(Y^0, R\right) \text{ : working ball}$$

$$\text{Behavior at } \infty \text{ : } \tilde{\tilde{J}}_i \to \infty \text{ as } \|Y\| \to \infty.$$

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Properties of transformed criteria

For all i :

- J_i and $\tilde{\tilde{J_i}}$ have same regularity.
- $\tilde{\tilde{J_i}}$ is dimensionless and strictly positive, it varies as J_i itself in the working ball $\mathcal{B}_R = \mathcal{B}\left(Y^0, R\right)$;
- For appropriate α_i and γ: $\left\|\nabla \tilde{\tilde{J_i}}^{\,0}\right\| \sim 1$
- $\tilde{\tilde{J_i}}\left(Y^0\right) = 1$ and $\lim_{\|Y\|\to\infty} \tilde{\tilde{J_i}} = \infty$;

DOUBLE SUPERSCRIPT $\tilde{\tilde{}}$ IMPLICIT FROM HERE ON

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Extend notion of stationarity

Lemma : Let Y^0 be a Pareto-optimal point of the smooth criteria $J_i(Y)$ $(1 \le i \le n \le N)$, and define the gradient-vectors $u_i^0 = \nabla J_i\left(Y^0\right)$ in which ∇ denotes the gradient operator. There exists a convex combination of the gradient-vectors that is equal to zero:

$$\sum_{i=1}^{n} \alpha_i u_i^0 = 0\,, \qquad \alpha_i \ge 0 \ (\forall i)\,, \qquad \sum_{i=1}^{n} \alpha_i = 1\,.$$

Proposed definition :*[Pareto-stationarity]*
The smooth criteria $J_i(Y)$ $(1 \le i \le n \le N)$ are [here] said to be Pareto-stationary at the design-point Y^0 iff there exists a convex combination of the gradient-vectors, $u_i^0 = \nabla J_i(Y^0)$, that is equal to zero.

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Postulate of evidence

At Pareto-optimal design-points, we cannot improve all criteria simultaneously
... BUT AT ALL OTHER DESIGN-POINTS ... YES, WE CAN ![1]

In an optimization iteration, Nash equilibrium design-points should only be sought after completion of a cooperative-optimization phase during which all criteria improve.

[1] Obama, 2009

Descent direction common to n disciplines (1)

Lemma :

Let $\{u_i\}$ $(i = 1, 2, ..., n)$ be a family of n vectors in a Hilbert space $\mathcal{H}$ of dimension at least equal to n. Let U be the set of the strict convex combinations of these vectors:

$$U = \left\{ w \in H \,/\, w = \sum_{i=1}^{n} \alpha_i u_i\,;\ \alpha_i > 0\,,\ \forall i\,;\ \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

and $\overline{U}$ its closure, the convex hull of the family. Let ω be the unique element of $\overline{U}$ of minimal norm. Then :

$$\forall \bar{u} \in \overline{U},\ (\omega, \bar{u}) \geq \|\omega\|^2 := C_\omega \geq 0$$

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Descent direction common to n disciplines (2)

Proof of Lemma :

Existence and uniqueness of the minimal-norm element $\omega \in \overline{U}$:
$\overline{U}$ is closed and convex, $\| \; \|$ is continuous, and bounded from below.
Let $\bar{u} \in \overline{U}$ (arbitrary) and $r = \bar{u} - \omega$. Since $\overline{U}$ is convex :

$$\forall \varepsilon \in [0,1]\,,\; \omega + \varepsilon r \in \overline{U}$$

Since ω is the minimal-norm element $\in \overline{U}$:

$$\|\omega + \varepsilon r\|^2 - \|\omega\|^2 = (\omega + \varepsilon r, \omega + \varepsilon r) - (\omega, \omega) = 2\varepsilon(\omega, r) + \varepsilon^2 (r, r) \geq 0$$

and this implies that $(\omega, r) \geq 0$; in other words :

$$\forall \bar{u} \in \overline{U}\,,\; (\omega, \bar{u} - \omega) \geq 0$$

where equality stands whenever ω is the orthogonal projection of 0 onto $\overline{U}$. Etc.

$\square$

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Descent direction common to n disciplines (3)

Theorem :

Let $\mathcal{H}$ be a Hilbert space of finite or infinite dimension N. Let $J_i(Y)$ ($1 \leq i \leq n \leq N$) be n smooth functions of the vector $Y \in \mathcal{H}$, and Y^0 a particular admissible design-point, at which the gradient-vectors are denoted $u_i^0 = \nabla J_i(Y^0)$, and

$$\mathcal{U} = \left\{ w \in \mathcal{H} \;/\; w = \sum_{i=1}^{n} \alpha_i \, u_i^0 \,;\; \alpha_i > 0 \; (\forall i) \,;\; \sum_{i=1}^{n} \alpha_i = 1 \right\} \tag{23}$$

Let ω be the minimal-norm element of the convex hull $\overline{\mathcal{U}}$, closure of $\mathcal{U}$. Then :

1. either $\omega = 0$, and the criteria $J_i(Y)$ ($1 \leq i \leq n$) are Pareto-stationary at $Y = Y^0$;
2. or $\omega \neq 0$ and $-\omega$ is a descent direction common to all the criteria; additionally, if $\omega \in \mathcal{U}$, the inner product $(\bar{u}, \omega)$ is equal to the positive constant $C_\omega = \|\omega\|^2$ for all $\bar{u} \in \overline{U}$.

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Descent direction common to *n* disciplines (4)

Proof of Theorem :

The first part of the conclusion is a direct application of the Lemma.

Directional derivatives : $\{(u_i,\omega)\}$ $(i=1,2,...,n)$.
Assume that $\omega \in U$ and not simply $\overline{U}$.
Define $\mathrm{j}(u) = \|u\|^2 = (u,u)$. Then, ω is the solution to the following minimization problem :

$$\min_{\alpha} \mathrm{j}(u)\,,\ u = \sum_{i=1}^{n} \alpha_i u_i\,,\ \sum_{i=1}^{n} \alpha_i = 1$$

since none of the constraints $\alpha_i \geq 0$ is saturated. The Lagrangian,

$$h = \mathrm{j} + \lambda \left(\sum_{i=1}^{n} \alpha_i - 1 \right)$$

is stationary w.r.t the vector $\alpha \in \mathbb{R}_+^N$ and the real variable λ :

$$\forall i\ :\ \frac{\partial h}{\partial \alpha_i} = 0\,,\ \text{et}\ \frac{\partial h}{\partial \lambda} = 0$$

Therefore, for any index i :

$$\frac{\partial \mathrm{j}}{\partial \alpha_i} + \lambda = 0$$

But, $\mathrm{j}(u) = (u,u)$ and for $u = \omega = \sum_{i=1}^{n} \alpha_i u_i$, we have:

$$\frac{\partial \mathrm{j}}{\partial \alpha_i} = 2(\frac{\partial u}{\partial \alpha_i}, u) = 2(u_i,\omega) = -\lambda \Longrightarrow (u_i,\omega) = -\lambda/2 \text{ (a constant).}$$

By linearity, this extends to any convex combination of the $\{u_i\}_{(i=1,2,...,n)}$. $\square$

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"Cooperative-Optimization" : Multiple-Gradient Descent Algorithm (MGDA)

From a non-stationary design-point Y^0, construct a sequence $\{Y^i\}$ $(i = 0, 1, 2...)$:

Compute for all i $(1 \leq i \leq n)$:

$$u_i^0 = \nabla J_i^0$$

and apply the theorem to define ω^0. If $\omega^0 \neq 0$, consider:

$$j_i(t) = J_i\left(Y^0 - t\omega^0\right) \quad (1 \leq i \leq n)$$

and identify $h^0 > 0$, the largest real number for which these functions of t are strictly-monotone decreasing over $[0, h^0]$. Let:

$$Y^1 = Y^0 - h^0\,\omega^0$$

so that:

$$J_i\left(Y^1\right) < J_i\left(Y^0\right)$$

and so on.

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Two possible situations

Either: the construction stops after a finite number of steps, at a P-stationary design-point Y^r; then possibly proceed with the "competitive-optimization" phase;

or: the sequence $\{Y^i\}$ is infinite.

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Case of an infinite sequence $\{Y^i\}$ $(i = 0, 1, 2...)$

Then:

- The corresponding sequence of criterion $\{J_i\}$, for any given i, is strictly monotone-decreasing, and positive, thus bounded.
- Since the criterion $J_i(Y)$ is ∞ at ∞, the sequence $\{Y^i\}$ is itself bounded. ($\mathcal{H}$ is assumed reflexive.)
- There exists a weakly convergent subsequence; let Y^* be the limit.

We conjecture that Y^* is P-stationary.
(Otherwise, restart with $Y^0 = Y^*$.)

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Summary : practical implementation

One is led to solve the following quadratic-form minimization in $\mathbb{R}^n$:

$$\min_{\alpha \in \mathbb{R}^n} \|\omega\|^2$$

subject to the following constraints/notations :

$$\omega = \sum_{i=1}^{n} \alpha_i u_i \,,\; u_i = \nabla J_i \left(Y^0\right) \,,\; \alpha_i \geq 0 \;(\forall i) \,,\; \sum_{i=1}^{n} \alpha_i = 1$$

Then, we recommend :

- if $\omega \neq 0$, *to use $-\omega$ as a descent direction;*
- otherwise *(Pareto-stationarity)*, to analyze local Hessians, and :
    - if all positive-definite *(Pareto-optimality)*: *stop;*
    - otherwise : *stop anyway (if design satisfactory), or elaborate a sensible Nash game from Y^0 in the eigenvector basis of $\sum_{i=1}^{n} \alpha_i H_i^0$.*

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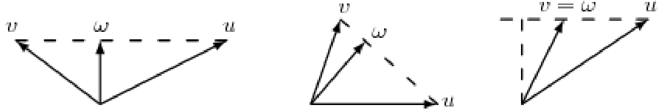
Cooperative phase

Let:
$u = u_1 = \nabla J_1\left(Y^0\right)$, $v = u_2 = \nabla J_2\left(Y^0\right)$, $\alpha_1 = \alpha$, $\alpha_2 = 1 - \alpha$.
Then :

$$\alpha^* = \frac{v.(v-u)}{\|u-v\|^2} = \frac{\|v\|^2 - v.u}{\|u\|^2 + \|v\|^2 - 2u.v}$$

$$0 < \alpha^* < 1 \iff \widehat{(u,v)} > \cos^{-1} \frac{\min\left(\|u\|, \|v\|\right)}{\max\left(\|u\|, \|v\|\right)}$$



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Competitive phase

What to do if the initial design-point Y^0 is Pareto-stationary w.r.t. $(J_A\,,\ J_B)$?

Let us examine first the convex case:

- Stationary point of type I : $\nabla J_A^0 = \nabla J_B^0 = 0$
  Simultaneous minimum of J_A and J_B: STOP
- Stationary point of type II : e.g. $\nabla J_A^0 = 0$ and $\nabla J_B^0 \neq 0$
  J_A minimum, J_B reducible: STOP, or
  NASH equilibrium with hierarchical split of variables
- Stationary point of type III : $\nabla J_A^0 + \lambda \nabla J_B^0 = 0$ $(\lambda > 0)$
  Pareto-optimality: STOP

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Non-convex case (1)

P-Stationary design-point of type I : $\nabla J_A^0 = \nabla J_B^0 = 0$

H_A^0, H_B^0: Hessian matrices of J_A, J_B at $Y = Y^0$

- If $H_A^0 > 0$ and $H_B^0 > 0$: CONVEX CASE: STOP
- $H_A^0 > 0$ and H_B^0 has some <0 eigenvalues
  J_A minimum, J_B is reducible:
  STOP, or NASH equilibrium with the hierarchical split of territory based on the eigenstructure of the Hessian matrix H_A^0.

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Non-convex case (2)

P-Stationary design-point of type I : $\nabla J_A^0 = \nabla J_B^0 = 0$

- If both Hessian matrices have some <0 eigenvalues, define families of linearly independent eigenvectors:

$$\mathcal{F}_A = \{ u_1 \,,\, u_2 \,,\, \dots \,,\, u_p \} \qquad \mathcal{F}_B = \{ v_1 \,,\, v_2 \,,\, \dots \,,\, v_q \}$$

  - If $\mathcal{F}_A \cup \mathcal{F}_B$ is linearly dependent, $\sum_{i=1}^{p} \alpha_i \, u_i - \sum_{j=1}^{q} \beta_j \, v_j = 0$
    Then, a common descent direction is $-w^r$:

$$w^r = \sum_{i=1}^{p} \alpha_i \, u_i = \sum_{j=1}^{q} \beta_j \, v_j$$

  - Otherwise, $Sp\mathcal{F}_A \cap Sp\mathcal{F}_B = \{0\}$: STOP, OR determine the NASH equilibrium point using $\mathcal{F}_A$ (resp. $\mathcal{F}_B$) as the strategy of A (resp. B).

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Non-convex case (3)

P-Stationary design-point of type II : $\nabla J_A^0 = 0$ and $\nabla J_B^0 \neq 0$

- $H_A^0 > 0$:
  Case already studied: NASH equilibrium in the hierarchical basis of eigenvectors of H_A^0.
- H_A^0 has some <0 eigenvalues associated with the eigenvectors: $\mathcal{F}_A = \{ u_1 , u_2 , \ldots , u_p \}$
  - if ∇J_B^0 is not $\perp Sp\mathcal{F}_A$: a descent direction common to J_A and J_B exists in $Sp\mathcal{F}_A$: use it to reduce both criteria.
  - otherwise, $\nabla J_B^0 \perp Sp\mathcal{F}_A$: we propose to identify the NASH equilibrium using same split as above.

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Non-convex case (4)

P-Stationary design-point of type III :
$\nabla J_A^0 + \lambda \nabla J_B^0 = 0 \ (\lambda > 0)$

Let

$$u_{AB} = \frac{\nabla J_A^0}{\left\|\nabla J_A^0\right\|} = -\frac{\nabla J_B^0}{\left\|\nabla J_B^0\right\|}$$

Consider possible move in hyperplane $\perp u_{AB}$.
For this, consider reduced Hessian matrices:

$$H_A'^{\,0} = P_{AB}\, H_A^0\, P_{AB} \qquad H_B'^{\,0} = P_{AB}\, H_B^0\, P_{AB}$$

where: $P_{AB} = I - [u_{AB}]\,[u_{AB}]^t$.
Analysis in orthogonal hyperplane is that of a stationary point of type a and dimension $N-1$.

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Recommended strategy for multidisciplinary optimization

Design of Experiment

Select an appropriate set of initial designs

For each initial design :

- Perform a "COOPERATIVE-OPTIMIZATION' phase : at each iteration, all criteria improve
- Stop, or enter a "COMPETITIVE-OPTIMIZATION" phase :
  - perform an eigen-analysis of local systems,
  - define an appropriate split of variables, and
  - establish the corresponding Nash equilibrium between disciplines by SMOOTH CONTINUATION

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